

## Problem Set 8

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Three Questions due November 14, 2007

**1. Bargaining.** It is November 1st, and Angus and Bronwen are arguing over how to divide the remainder of their left-over trick or treat candy. They decide on the following rules. On Nov 1st, Angus will make Bronwen an offer  $[s_A(1), s_B(1)]$  where  $s_A(1) + s_B(1) = 1$ . If Bronwen accepts then Angus gets a share  $s(1)$  of the candy and Bronwen gets  $s_B(1)$ . If Bronwen refuses, however, then no-one gets any candy on Nov 1st. In this case, on Nov 2nd, Bronwen gets to make Angus an offer  $[s_A(2), s_B(2)]$  where  $s_A(2) + s_B(2) = 1$ . If Angus accepts then Angus gets a share  $s_A(2)$  of the candy and Bronwen gets  $s_B(2)$ . If Angus refuses, however, then no-one gets any candy on Nov 2nd. In this case, on Nov 3rd, Angus gets to make Bronwen an offer  $[s_A(3), s_B(3)]$  where  $s_A(3) + s_B(3) = 1$ . If Bronwen accepts then Angus gets a share  $s_A(3)$  of the candy and Bronwen gets  $s_B(3)$ . If Bronwen refuses, however, then no-one gets any candy on Nov 3rd. In this case, on Nov 4th, Bronwen gets to make Angus an offer  $[s_A(4), s_B(4)]$  where  $s_A(4) + s_B(4) = 1$ . If Angus accepts then Angus gets a share  $s_A(4)$  of the candy and Bronwen gets  $s_B(4)$ . If Angus refuses, however, then all the candy is thrown away by their parents.

There is a catch. Both Angus and Bronwen are hungry and impatient for candy so they discount candy tomorrow. For Angus, each day, one candy tomorrow is worth only  $\delta_A$  candy today where  $\delta_A < 1$ . For Bronwen, each day, one candy tomorrow is only worth only  $\delta_B$  candy today where  $\delta_B < 1$ . It is commonly known that Angus and Bronwen are rational and that they are not interested in distributive justice. You may also assume that each of Angus and Bronwen accepts an offer whenever he or she is indifferent.

(a) What do you expect to happen, and what allocation of candy would arise if  $\delta_A = \delta_B$ ? Explain your reasoning carefully.

(b) Compare your answer when  $\delta_A = \delta_B = \frac{3}{4}$ , and when  $\delta_A = \delta_B = \frac{1}{2}$ . Intuitively, why is the impatience of both good for one and bad for the other.

(c) Now suppose that Angus becomes more impatient so that  $\delta_A < \delta_B$ . Does this change increase or decrease Angus's share of the candy relative to what it was when  $\delta_A = \delta_B$ . Give an intuition.

**2. Information and Nuclear Safety.** Consider the following game involving two real players and a chance move by 'nature'. America and Russia have the nuclear capability to destroy each other. 'Nature' tosses a fair coin so that with probability  $\frac{1}{2}$  America moves first and Russia moves second, and with probability  $\frac{1}{2}$  Russia moves first and America moves second. For now, assume that both countries observe nature's choice so they know whether they are first or second. The country who moves first decides whether to 'fire' its missiles or to 'wait'. If it fires, the

game ends: the country who fired gets a payoff of  $-1$ , and the other country gets  $-4$ . If the first country waits, then the second country gets to move. It too must decide to ‘fire’ or to ‘wait’. If it ‘fires’ then the game ends, it gets  $-1$  and the other country gets  $-4$ . If it ‘waits’ then both countries get  $0$ . Assume that each country maximizes its expected payoff.

Treat this as one game, rather than as two different games. Figure 0.1 is an extensive form (game tree) representing this game. The first payoff refers to America. The second payoff refers to Russia. There are no payoffs for ‘nature’.

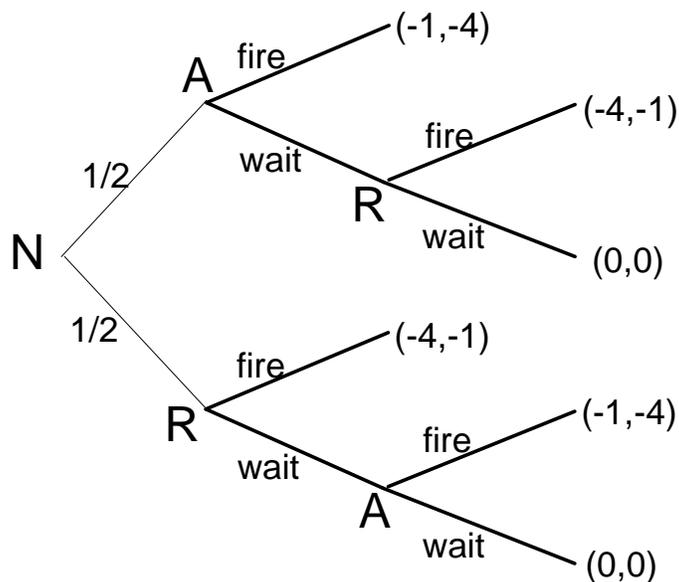


Figure 0.1: Nuclear Threat Game: Perfect Information

- What makes this a game of *perfect information*? Write down the definition of a strategy in an extensive form game, and identify the possible strategies for America and for Russia in this game.
- Find and explain any pure-strategy subgame-perfect equilibria (SPE), making clear what constitutes a subgame. Are there any Nash equilibria which are not SPE?
- Now suppose that neither Russia or America observes the move by nature, or each other’s move. That is, should a country be called upon to move, it does not know whether it is the first mover or whether it is the second mover and the other country chose ‘wait’. Again, treat this as one game. Draw a game tree similar to figure 0.1 but for this new game. Indicate clearly which nodes are in the same information sets.
- Identify the possible strategies for America and for Russia in the game from part (c). Find and explain carefully two ‘symmetric’ pure-strategy SPEs in this game that have very different outcomes.

- (e) Now suppose that America can observe the move by nature and also (when it is the second mover) Russia's move. Russia knows what America can observe, but, as before, Russia can observe neither nature's nor America's move. Draw the game tree for this game. Argue whether you think the world is a safer place or a more dangerous place now that America is better informed than Russia. That is, compare the SPE of this game with the SPE of the game of parts (b) and (d).

**3. Congestion and Toll Booths.** Sometimes very long queues build up at toll booths. You find yourself, say, waiting 5 minutes to pay 50¢. An often heard 'kvetch' is that such tolls are the cause of long commutes and should be eliminated. This question asks you to apply some game theory to this problem. In effect, you are going to design tolls that minimize traffic problems! The whole question is quite hard, but each part is not too bad so do not panic: just take it a step at a time.

Suppose that a very large (say, infinite) number of commuters must travel, one per car, from  $A$  to  $B$ . There are two possible routes, 1 and 2. Let  $x$  be the proportion of cars that use route 1. There is a public debate over whether to put a toll on route 1. The time it takes each car to get from  $A$  to  $B$ , depends on the proportion of other cars on that route. The time, in hours, each car takes on Route 1 is given by

$$T_1(x) = \begin{cases} 3.8x & \text{if there is no toll} \\ 4x & \text{if there is a toll} \end{cases}$$

The time each car takes on route 2 is given by

$$T_2(x) = 1 + 2(1 - x)$$

The utility of a commuter who chooses route 2 is  $-T_2(x)$ . Her utility if she chooses route 1 is  $-T_1(x)$  if there is no toll, and  $-T_1(x) - p$  if there is a toll of  $\$p$ . In principle,  $p$  could be negative.

(a) Consider the game among the commuters simultaneously choosing which route to take. Find the Nash-equilibrium proportion  $x$  that choose route 1 when there is no toll. How long does the commute take each car on route 1, and on route 2? [Hint: in equilibrium, what must be true of the utilities of a commuter who chooses route 1 and one who chooses route 2? You might want to use a calculator.]

(b) Find the Nash-equilibrium  $x$  when there is a toll, as a function of the toll charge  $p$ .

(c) Consider a social planner who can set  $x$  and who aims to minimize average commuting time  $\mathcal{T}(x) := xT_1(x) + (1-x)T_2(x)$ . (We assume that the social planner does not care about the money that changes hands at the toll booth.) Find the social planner's optimum  $x$  when there is a toll. What are the corresponding commutes on each route, and the average commuting time  $\mathcal{T}$ ?

(d) Suppose that the social planner can not direct the traffic  $x$ , but she can set the toll charge  $p$ . She could also choose no toll. Consider the following game. First, the social planner sets the toll (or chooses no toll). Then, after observing the toll charge (or no toll), the commuters simultaneously choose their routes (much as in parts (a) and (b)). Consider the subgame perfect

equilibrium outcome of this game. Does the social planner set a toll and, if so, what toll does she choose? [Hint: from part (b), you already know the relation between  $p$  and  $x$  if there is a toll. From part (c), if there is to be a toll, you already know the  $x$  the social planner would like to choose, and the  $\mathcal{T}$  that this induces. And, from part (a), you can figure out the  $\mathcal{T}$  without a toll.]

(e) Assuming that the time spent at the toll booth is  $0.2x$ , in the subgame-perfect equilibrium outcome from part (d), how long will each car on route 1 spend at the booth?