Problem Set 8 Solutions

1. *Find the real part, imaginary part, modulus, complex conjugate, and inverse of the following numbers: (i)* $\frac{2}{3+4i}$*, (ii)* $(3+4i)^2$*, (iii)* $\frac{3+4i}{3-4i}$*, (iv)* $\frac{1+\sqrt{2}i}{1-\sqrt{3}i}$*, and (v)* $\cos\theta + i\sin\theta$*.*

   To find the quantities we are looking for, we need to put the complex number into the form $z = a + bi$. Then, the modulus is $|z| = \sqrt{a^2+b^2}$, the complex conjugate is $z^* = a - bi$, and the inverse can be found using the previous quantities, as shown below

$$z^{-1} = \frac{1}{z} = \frac{1}{z}\left(\frac{z^*}{z^*}\right) = \frac{z^*}{|z|^2}$$

(i)

$$z = \frac{2}{3+4i} = \frac{2}{3+4i}\left(\frac{3-4i}{3-4i}\right) = \frac{2}{25}(3-4i) \implies Re(z) = \frac{6}{25},\ \ Im(z) = -\frac{8}{25}$$

$$|z| = \frac{2}{25}\sqrt{3^2+4^2} = \frac{2}{5}$$
$$z^* = \frac{2}{25}(3+4i)$$
$$z^{-1} = \frac{1}{2}(3+4i)$$

(ii)

$$z = (3+4i)^2 = -7+24i \implies Re(z) = -7,\ \ Im(z) = 24$$
$$|z| = \sqrt{7^2+24^2} = 25$$
$$z^* = -7-24i$$
$$z^{-1} = -\frac{1}{25^2}(7+24i) = -\frac{1}{625}(7+24i)$$

(iii)

$$z = \frac{3+4i}{3-4i} = \frac{3+4i}{3-4i}\left(\frac{3+4i}{3+4i}\right) = \frac{1}{25}(-7+24i) \implies Re(z) = -\frac{7}{25},\ \ Im(z) = \frac{24}{25}$$

$$|z| = \frac{1}{25}\sqrt{7^2+24^2} = 1$$
$$z^* = -\frac{1}{25}(7+24i)$$
$$z^{-1} = -\frac{1}{25}(7+24i)$$

(iv)

$$z = \frac{1+\sqrt{2}i}{1-\sqrt{3}i} = \frac{1+\sqrt{2}i}{1-\sqrt{3}i}\left(\frac{1+\sqrt{3}i}{1+\sqrt{3}i}\right) = \frac{1}{4}\left((1-\sqrt{6}) + i(\sqrt{2}+\sqrt{3})\right) \implies Re(z) = \frac{1-\sqrt{6}}{4},\ \ Im(z) = \frac{\sqrt{2}+\sqrt{3}}{4}$$

$$|z| = \frac{1}{4}\sqrt{(1-\sqrt{6})^2+(\sqrt{2}+\sqrt{3})^2} = \frac{\sqrt{12}}{4} = \frac{\sqrt{3}}{2}$$
$$z^* = \frac{1}{4}\left((1-\sqrt{6}) - i(\sqrt{2}+\sqrt{3})\right)$$
$$z^{-1} = \frac{1}{3}\left((1-\sqrt{6}) - i(\sqrt{2}+\sqrt{3})\right)$$

(v)

$$\begin{aligned}
z = \cos\theta + i\sin\theta \implies Re(z) = \cos\theta,\ \ Im(z) = \sin\theta \\
|z| = \sqrt{cos^2\theta + \sin^2\theta} = 1 \\
z^* = \cos\theta - i\sin\theta \\
z^{-1} = \cos\theta - i\sin\theta
\end{aligned}$$

2. *For the following pair of numbers, give their polar form, their complex conjugates, their moduli, product, the quotient $z1/z2$, and the complex conjugate of the quotient.*

$$z_1 = \frac{1+i}{\sqrt{2}} \qquad z_2 = \sqrt{3} - i$$

The polar form of a complex number is given by $z = Ae^{i\theta}$ where $A = |z|$ and $\theta = \tan^{-1}\frac{Im(z)}{Re(z)}$. Using this,

$$\begin{aligned}
z_1 &= e^{i\pi/4} & z_2 &= 2e^{-i\pi/6} \\
z_1^* &= \frac{1-i}{\sqrt{2}} = e^{-i\pi/4} & z_2^* &= \sqrt{3} + i = 2e^{i\pi/6} \\
|z_1| &= 1 & |z_2| &= 2
\end{aligned}$$

$$\begin{aligned}
z_1 * z_2 &= 2e^{i(\pi/4 - \pi/6)} = 2e^{i\pi/12} \\
\frac{z_1}{z_2} &= \frac{1}{2}e^{i(\pi/4+\pi/6)} = \frac{1}{2}e^{i5\pi/12} \\
\left(\frac{z_1}{z_2}\right)^* &= \frac{1}{2}e^{-i5\pi/12}
\end{aligned}$$

3. *Express the sum of the follwing in polar form:*

$$z_1 = 2e^{i\pi/4} \qquad z_2 = 6e^{i\pi/3}$$

$$\begin{aligned}
z_1 + z_2 &= 2(\cos\pi/4 + i\sin\pi/4) + 6(\cos\pi/3 + i\sin\pi/3) \\
&= 2\left(\frac{1}{\sqrt{2}}\right)(1+i) + 6\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\
&= \left(\sqrt{2} + 3\right) + i\left(\sqrt{2} + 3\sqrt{3}\right) \\
&= \sqrt{40 + \frac{12}{\sqrt{2}}(1+\sqrt{3})}e^{i\tan^{-1}\frac{\sqrt{2}+3\sqrt{3}}{\sqrt{2}+3}} \\
&= 7.95e^{i0.982}
\end{aligned}$$

4. *Consider De Moivre's Theorem, which states that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. This follows from taking the n-th power of both sides of Euler's theorem. Find the formula for $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$. Given $e^{iA}e^{iB} = e^{i(A+B)}$ deduce $\cos(A+B)$ and $\sin(A+B)$.*

   We can find the formula for $\cos 4\theta$ and $\sin 4\theta$ by expanding $(\cos\theta + i\sin\theta)^4$ and matching the real and imaginary parts to $\cos 4\theta$ and $\sin 4\theta$ respectively.

$$\begin{aligned}
(\cos\theta + i\sin\theta)^4 &= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta \\
&= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + 4i(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta) \\
&= \cos 4\theta + i\sin 4\theta
\end{aligned}$$

   Matching the real and imaginary parts we see that

$$\begin{aligned}
\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\
\sin 4\theta &= 4\cos\theta\sin\theta(\cos^2\theta - \sin^2\theta).
\end{aligned}$$

5. *Consider a particle attached to a spring executing a motion $x = A\sin(\omega t + \delta)$ with $A = .32\ m$. At $t = 0$, it is at $x = -.07m$ and a velocity $-2m/s$. The total energy is $5.6J$. Find (i) δ, (ii) f the frequency, (iii) k and (iv) m.*

   (i) To find δ, look at the equation of motion for x at $t = 0$.

   $$x(t=0) = x_0 = A\sin(0+\delta) \implies \delta = \sin^{-1}\left(\frac{x_0}{A}\right) = \sin^{-1}\left(\frac{-0.07}{0.32}\right).$$

   The arcsine of a number has two possible values becuase in a 2π period sine takes on all values between -1 and 1 twice. In this case, we also know that we want $\cos\delta$ to be negative becuase the velocity is negative at $t = 0$. (If we choose the other value for $\sin^{-1}\delta$ then we will get a negative ω. This is fine, it just means that our system is out of phase by π.)
   In this case, $\sin^{-1}\left(\frac{-0.07}{0.32}\right) = -0.221$; however, $2.92 = -0.221 + \pi$ is also a valid answer. The second answer is in the third quadrent which is where both sine and cosine are negative, so I will take $\delta = 2.92$.

   (ii) To find the frequency f, we need to look at the equation for the velocity. Differentiating $x(t)$ we find, $v(t) = A\omega\cos(\omega t + \delta)$. Evaluating $v(t)$ at $t = 0$ we find,

   $$v(0) = v_0 = A\omega\cos(0+\delta) \implies \omega = \frac{v_0}{A\cos\delta}.$$

   The frequency, f, is given by $\omega/2\pi$, so

   $$f = \frac{v_0}{2\pi A\cos\delta} = \frac{-2\,m/s}{2\pi(0.32m)\cos(2.92)} = 1.01\ cycles/sec.$$

   (iii) When the spring is stretched to its maximum length, all of the energy in the system is potential energy, therefore, $E = \frac{1}{2}kx_{max}^2$. The maximum distance occurs when $|\sin(\omega t + \delta)| = 1$.

   $$E = \frac{1}{2}kx_{max}^2 = \frac{1}{2}kA^2$$
   $$5.6\,J = \frac{1}{2}k(0.32\,m)^2 \implies k = 109\,N/m$$

   (iv) The frequency of a spring is given by $\omega = \sqrt{\frac{k}{m}}$. Using this formula,

   $$m = \frac{k}{\omega^2} = \frac{109\,N/m}{(2\pi(1.02))^2\,sec^{-2}} = 2.7\,kg.$$

6. *A mass m moving horizontally at velocity v_0 on a frictionless table strikes a spring of force constant k. It compresses the spring and then bounces back with opposite velocity. Assuming no loss of energy anywhere find out (i) how long the mass is in contact with the spring and (ii) the maximum compression of the spring.*

   (i) The frequency ω for a mass oscillating on a spring is given by $\omega = \sqrt{\frac{k}{m}}$. The period of an oscillation is then $T = \frac{2\pi}{\omega}$. In this problem, the mass hits the spring at $x = 0$, compresses it, bounces back to $x = 0$, and then leaves the spring. Therefore, the mass is in contact with the spring for half of a period. (We assume the spring is massless, so it does not continue to stretch once the mass passes $x = 0$.) The total time t the mass is in contact with the spring is

   $$t = \frac{T}{2} = \pi\sqrt{\frac{m}{k}}.$$

   (ii) We can find the maximum compression of the spring by conservation of energy. When the mass just hits the spring, all the energy is kinetic. At the maximum compression of the spring, all the energy is potential energy. Therefore,

   $$\frac{1}{2}mv_0^2 = \frac{1}{2}kA^2 \implies A = v_0\sqrt{\frac{m}{k}}$$

   where A is the maximum compression of the spring.

7. *A steel beam of mass M and length L is suspended at its midpoint by a cable and executes torsional oscillations. If two masses m are now attached to either end of the beam and this reduces the frequency by 10%, what is m/M?*

   The frequency ω is proportional to $I^{-1/2}$ where I is the moment of inertia. The moment if inertia of just the steel beam is $I_i = ML^2/12$. Once you add the two masses at either end, the moment of inertia becomes $I_f = ML^2/12 + 2m(L/2)^2 = ML^2/12 + mL^2/2$. Taking the ratio of the ω_f to ω_i,

$$\begin{aligned}
\frac{\omega_f}{\omega_i} &= \sqrt{\frac{I_i}{I_f}} \\
0.9 &= \sqrt{\frac{\frac{1}{12}ML^2}{\frac{1}{12}ML^2 + \frac{1}{2}mL^2}} \\
(0.9)^2 &= \frac{1}{1 + 6\frac{m}{M}} \\
\frac{m}{M} &= \frac{1}{6}\left(\frac{1}{0.9^2} - 1\right) = 0.039.
\end{aligned}$$

   The masses on the ends are each about 4% of the mass of the beam.

8. *Imagine a solid disc, (say a penny), of mass M, radius R, standing vertically on a table. A tiny mass m of negligible size is now glued to the rim at the lowest point. When disturbed, the penny rocks back and forth without slipping. Show that the period of the Simple Harmonic Motion is*

$$T = 2\pi\sqrt{\frac{3MR}{2mg}}.$$

   Hint: Find κ, the restoring torque per unit angular displacement. When $m \to 0$ what happens to T. Explain in physical terms.

   I will first compute the period starting from $\sum \tau = I\alpha$. Take the pivot point to be the point on the table a distance $R\theta$ away from $x = 0$, where $x = 0$ is the point of contact between the penny and the table before the oscillations. When the penny has rotated to $x = R\theta$, as shown in Figure 1, the only force not acting at the pivot is the force of gravity on the mass m. The moment of interia about the pivot point is $I = I_{penny} + I_m$ where $I_{penny} = \frac{3}{2}MR^2$ by the parallel axis theorm and $I_m = m(R\theta)^2$ plus terms that depend on higher powers of θ. Because θ must be small in order to have simple harmonic motion, we will only keep terms that are proportional to θ, but not θ^2 or any higher powers. Therefore, $I = I_{penny}$ and

$$\begin{aligned}
\sum \tau &= I\alpha \\
mg(R\theta) &= -\frac{3}{2}MR^2\ddot{\theta} \\
\frac{2mg}{3MR}\theta &= -\ddot{\theta}.
\end{aligned}$$

   Comparing to the equation for simple harmonic motion, $\omega^2 x = -\ddot{x}$, it is clear that $\omega^2 = \frac{2mg}{3MR}$. The period is given by $T = \frac{2\pi}{\omega}$, so as desired

$$T = 2\pi\sqrt{\frac{3MR}{2mg}}.$$

   We could also choose to calculate the torques about the point where m is attached to the penny. For small θ, this bottom point does not move when the coin rocks back and forth, so this is a reasonable point about which to measure torques. As shown in Figure 1, the two forces that do not act at the pivot point are the weigh Mg of the penny and the normal force $N = (m + M)g$ acting up from the table, where the pivot is the position of the mass m. Both these forces act a perpendicular distance of $R\theta$ away from the pivot point. The moment of

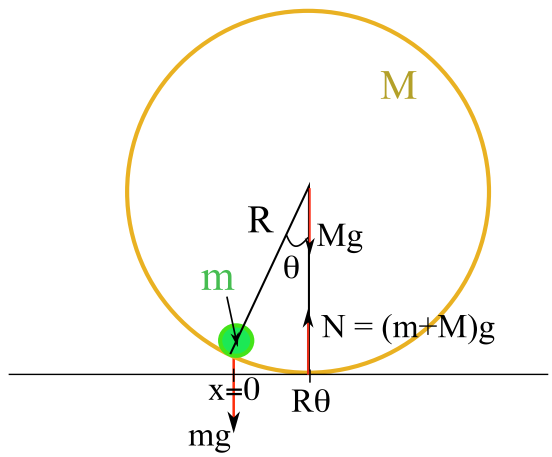

FIG. 1: The penny with a little mass m at the bottom for problem 8. The penny is rocking back and forth about the point $x = 0$. For small θ, the mass m is approximately stationary and does not move from $x = 0$. The forces acting on the penny are shown in red. (There is also the force due to friction which is not shown, but it never exerts a torque in the calculation.)

inertia of the disk about the pivot point is $\frac{3}{2}MR^2$ by the parallel axis theorem. Putting this all together,

$$\tau = I\alpha = \vec{r} \times \vec{F}$$
$$-\frac{3}{2}MR^2\ddot{\theta} = R\theta\left((m+M)g - Mg\right)$$
$$\ddot{\theta} = -\frac{2mg}{3MR}\theta$$

which is the same result we derived above.

You can also do this problem using energy. Taking the zero of the potential energy to be at the center of the disk, the potential energy is given by $U = -mgR(1-\cos\theta)$. To lowest order in θ this is $mgR\theta^2/2$. ($\cos\theta = 1-\theta^2/2+ ...$) Since the mass m is essentially stationary, the kinetic energy is given only by the rotation and translational motion of the penny. Thus the total energy is

$$E = mgR(\frac{1}{2}\theta^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}M(R\dot{\theta})^2$$
$$E = \frac{1}{2}(mgR)\,\theta^2 + \frac{1}{2}\left(\frac{3}{2}MR^2\right)\dot{\theta}^2$$

where $I = \frac{1}{2}MR^2$ for a disk rotating about its center. Comparing to the energy of a spring, which is $E = \frac{1}{2}kx^2 + \frac{1}{2}m_s v^2$, we can recognize that for this system $k = mgR$ and $m_s = \frac{3}{2}MR^2$. The frequency of oscillation is $\sqrt{\frac{k}{m_s}}$ which in this case is $\sqrt{\frac{2mg}{3MR}}$. This is exactly what we found using torques.

When $m = 0$, $T \to \infty$. An infinite period means the penny will never return to where it started, which makes sense. If you start the penny rolling without a mass attached to the bottom, it will just keep rolling in the same direction, not oscillate about the bottom point.

9. *I am driving my car on a parkway which has bumps every* $30\ m$ *apart. At what speed must I be driving to experience violent shaking if the suspension in my car has a resonant frequency of* $0.5\ Hz$*?*

   This problem is asking at what speed do you have to drive the car such that you hit the bumps in the road with the same frequency as the resonant frequency of the car. You want to hit the bumps with a frequency of $0.5\,Hz = 0.5\,sec^{-1}$, or one bump every 2 seconds.

$$f = \frac{1}{2}\,Hz = \frac{v}{d} = \frac{v}{30\,m} \implies v = 15\,m/s$$

   You have to drive at $15\,m/s$ or $33.5\,mph$.

For following problems the symbols are defined the following equations

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0\cos(\omega t) \tag{1}$$

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{2}$$

10. *Show that a driven oscillator has its maximum amplitude of vibration at a frequency* $\omega = \sqrt{\omega_0^2 - (b^2/2m^2)}$. *At what frequency does the velocity have the greatest amplitude?*

Equation 1 gives the equation of motion for a driven oscillator with damping. What we are looking for is the long time or steady state solution of this differential equation. From class, you know this solution takes the form

$$x(t) = x_0\cos(\omega t - \phi)$$

where

$$x_0 = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (b\omega)^2}} = \frac{F_0/m}{\sqrt{(w^2 - w_0^2)^2 + \gamma^2\omega^2}},$$

$\omega_0^2 = \frac{k}{m}$, and $\gamma = b/m$. x_0 will be maximized when the denominaor is minimized. We can find the minimum by taking $d/d\omega$ of the denominator and setting it equal to zero. Because of the square root, this looks mathematically complicated, but the square of a function has the same maxima and minima as the original functon, so, I will find the minimum of the denominator squared.

$$\begin{aligned}\frac{d}{d\omega}\left((k - m\omega^2)^2 + (b\omega)^2\right) &= 2(k - m\omega^2)(2m\omega) + 2b^2\omega \\ 0 &= 4km - 4m^2\omega^2 + 2b^2 \\ \omega^2 &= \frac{k}{m} + \frac{b^2}{2m^2} \\ \omega &= \sqrt{\omega_0^2 - \frac{b^2}{2m^2}}.\end{aligned}$$

As desired, this is the frequency where we see the maximum amplitude of vibration.

The velocity is proportional to the same denominator, but you get an extra factor of ω in the numerator from taking the derivative of $x(t)$ with respect to t. So, we need to maximize

$$\frac{w}{\sqrt{(k - m\omega^2)^2 + (b\omega)^2}}$$

to find the maximum amplitude for the velocity. It again will be much easier to maximize the square of this quantity, so squaring, differentiating with respect to ω, and setting the result equal to zero, we find that when

$$\omega = \sqrt{\frac{k}{m}} = \omega_0$$

the amplitude of the velocity is a maximum.

11. *For a damped oscillator (not driven by any external force) find the time* T^* *after which the amplitude of oscillations drops to half its value in terms of* b *and* m.

There are three types of damped oscillations - underdamped, overdampeed, and critically damped. As we will see, which one of these you have in a system depends on the values of m, b, and k. The differential equation for the damped oscillator with no driving force is given by

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0.$$

Taking $x(t) = Ae^{\alpha t}$ and plugging this into the differential equation,

$$\alpha^2 m + b\alpha + k = 0 \implies \alpha = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} = -\frac{\gamma}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2} \tag{3}$$

where γ and ω_0 are defined as before. If $b^2 > 4mk$ then α is real and the solutions are decaying exponentials. This is the overdamped case. If $b^2 < 4mk$ then α is complex and the solutions have oscillations which decay to zero. This is called underdamped. If $b^2 = 4mk$, the is the critically damped case. The underdamped case is the only time we get oscillations, so this is the case that is referred to in this problem.

The amplitude only depends on the real part of α. The imaginary part of α leads to oscillations in $x(t)$, but does not effect the amplitude. We want to know at what time T^* the amplitude will be half its initial value. $x(t = 0) = A$, so

$$x(T^*) = \frac{A}{2} = Ae^{Re[\alpha]T^*} = Ae^{-\frac{b}{2m}T^*} \implies T^* = \frac{2m}{b}\ln 2$$

Because the real part of α is the same in the critically damped case as in the underdamped case, this would also be the solution if the system were critically damped. For the overdamped case, α has two real solutions. This means that $x(t)$ has the form $x(t) = Ae^{\alpha_+ t} + Be^{\alpha_- t}$. In order to find the time T^*, we need more information about the initial conditions such that we can find A in terms of B.

12. *Consider a damped oscillator with $k = 32, m = .5, b = 1$ in MKS units.*
(i) Find the solution with $x(0) = 2, v(0) = 0$. I suggest using symbols till the every end.
(ii) Now add on a driving force $F_0 \cos\omega t$ with $F_0 = 10N$ and $\omega = 2\omega_0$. Find the solution with $x(0) = 2, v(0) = 0$.

(i) From the numbers given, we can clearly see that $b^2 - 4mk$ will be less than zero, so this oscillator will have an underdamped solution. From class, you know that the solution for an underdamped oscillator takes the form

$$x(t) = Ae^{-\gamma t/2}\cos(\omega' t - \phi_0)$$

where $\omega' = \frac{1}{2m}\sqrt{4mk - b^2} = \sqrt{\left(\frac{\gamma}{2}\right)^2 - w_0^2} = \sqrt{63} = 7.94$. Plugging in initial conditions we find,

$$x(0) = A\cos(-\phi_0) = 2 \implies A = \frac{2}{\cos\phi_0}$$
$$v(0) = -A\left(\frac{1}{2}\gamma\cos\phi_0 + \omega'\sin(-\phi_0)\right) = 0 \implies \phi_0 = 0.125.$$

Therefore, $A = 2.02$ and the full solution for the damped oscillator without a driving force is

$$x(t) = Ae^{-\gamma t/2}\cos(\omega' t - \phi_0) = 2.02e^{-t}\cos(7.94t - 0.125).$$

(ii) The full solution with a driving force is the solution we solved for above where the right hand side of the differential equation is zero (known as the complimentary solution) plus the solution that gives the right hand side equal to $F_0\cos\omega t$ (the particular solution). I will write this as $x(t) = x_c(t) + x_p(t)$ where $x_c(t)$ takes the form from part (i).
The particular solutions is the same solutions we used in problem 10,

$$x_p(t) = x_0\cos(\omega t + \phi) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}\cos(\omega t - \phi)$$

where $\omega_0 = \sqrt{\frac{k}{m}} = 8\,sec^{-1}$ and $\omega = 2\omega_0 = 16\,sec^{-1}$. We can find the value of ϕ by plugging back into the differential equation, given by Equation 1. Doing this we find that

$$\frac{F_0}{x_0 m}\cos\omega t = (\omega_0^2 - \omega^2)\cos(\omega t - \phi) - \gamma\omega\sin(\omega t - \phi).$$

This equation has to hold at all times, so choosing $t = 0$ to simplify things we have

$$\frac{F_0}{x_0 m} = (\omega_0^2 - \omega^2)\cos\phi + \gamma\omega\sin\phi$$
$$\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} = (\omega_0^2 - \omega^2)\cos\phi + \gamma\omega\sin\phi.$$

From this, we see that

$$\cos\phi = \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}, \qquad \sin\phi = \frac{\gamma\omega}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}.$$

Plugging in numbers we find, $\phi = 2.98$. We can also plug in numbers to calculate a value for x_0 and we find $x_0 = 0.103\,m$.
Now, we need to use the initial conditions to solve for the full solution

$$x(t) = x_p(t) + x_c(t) = x_0\cos(\omega t - \phi) + Ae^{-\gamma t/2}\cos(\omega' t - \phi_0)$$

where we have already solved for x_0, ϕ, and ω' above. At $t = 0$ we have

$$x(0) = 2 = x_0\cos\phi + A\cos\phi_0$$
$$v(0) = 0 = \omega x_0\sin\phi - A\left(\frac{1}{2}\gamma\cos\phi_0 - \omega'\sin\phi_0\right).$$

Solving these two equations for A and ϕ_0 we find that $A = 2.11\,m$ and $\phi_0 = 0.109$. Therefore, our full solution is

$$x(t) = 0.103\cos(16t - 2.98) + 2.11e^{-t}\cos(7.94t - 0.109)$$

where I have suppressed all the units.