Oscillator Notes.
Fall 2006 Physics 200a
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I will discuss here only what was left out near the end of the lecture of November 6 We were considering the solution to the driven oscillator

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0\cos(\omega t)$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

First note that even if $F_0 = 0$, there is a solution. (In class I referred to F_0 as simply F.) For the case $\gamma < 2\omega_0$, which we will focus on, it is

$$x_c(t) = Ce^{-\frac{\gamma t}{2}}\cos(\omega' t - \phi_0) \tag{1}$$

where

$$\omega' = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2} \qquad \gamma = \frac{b}{m} \tag{2}$$

and the subscript c stands for *complimentary solution*, the solution with no driving force. I have referred to the phase as ϕ_0 since another phase ϕ will enter shortly. The parameters C and ϕ_0 are found by demanding that the initial position and velocity, $x(0)$ and $v(0)$, have some prescribed values at $t = 0$.

Now consider the driven problem

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0\cos(\omega t) \tag{3}$$

which we rewrite as

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t). \tag{4}$$

Now introduce another problem where the driving force is $F_0\sin(\omega t)$ and the solution is $y(t)$:

$$\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega_0^2 y = \frac{F_0}{m}\sin(\omega t) \tag{5}$$

Now if we form

$$z(t) = x(t) + i\, y(t) \tag{6}$$

by adding the first equation 4 to i times the second, 5, we find

$$\frac{d^2z}{dt^2} + \gamma\frac{dz}{dt} + \omega_0^2 z = \frac{F_0}{m}e^{i\omega t}. \tag{7}$$

Our plan is to solve this equation for z and take the real part, which is our $x(t)$.

The nice thing about z is that we can guess a solution

$$z(t) = z_0 \exp^{i\omega t} \tag{8}$$

where z_0 is a constant (in time). Feeding in this guess (and remembering that every differentiation just brings down an $i\omega$, we find

$$(-\omega^2 + i\omega\gamma + \omega_0^2) z_0 \exp^{i\omega t} = \frac{F_0}{m} \exp^{i\omega t} \tag{9}$$

Cancelling the $e^{i\omega t}$ (which is never zero) we find that z_0 is fully determined by the above to be

$$z_0 = \frac{F_0/m}{(-\omega^2 + i\omega\gamma + \omega_0^2)} \equiv \frac{F_0/m}{I(\omega)} \tag{10}$$

where the impedance $I(\omega)$ is

$$I(\omega) = (-\omega^2 + i\omega\gamma + \omega_0^2). \tag{11}$$

Let us write I in polar form

$$I(\omega) = |I| e^{i\phi} \tag{12}$$

where

$$|I| = \sqrt{(-\omega^2 + \omega_0^2)^2 + (\omega^2\gamma^2)} \qquad \tan\phi = \frac{\omega\gamma}{(-\omega^2 + \omega_0^2)} \tag{13}$$

This means

$$z_0 = \frac{F_0}{m|I|e^{i\phi}} \tag{14}$$

and that

$$z(t) = \frac{F_0}{m|I|} e^{i(\omega t - \phi)} \tag{15}$$

and finally

$$x(t) = \frac{F_0}{m|I|} \cos(\omega t - \phi) \equiv x_0 \cos(\omega t - \phi) \tag{16}$$

(There is one more point about this answer I will return to shortly.)

Thus the driven oscillator vibrates at the frequency of the driving force, lags in phase by ϕ and has an amplitude $\frac{F_0}{m|I|}$. Both the amplitude and phase are frequency dependent.

The amplitude x_0 is largest where $|I|$ is smallest:

$$x_0 = \frac{F_0/m}{\sqrt{(-\omega^2 + \omega_0^2)^2 + (\omega^2\gamma^2)}} \tag{17}$$

If $\gamma = 0$, this clearly occurs at $\omega = \omega_0$. At this point x_0, the amplitude of vibrations diverges. This is however an un-physical case since there is always some friction or γ. In the presence of nonzero γ, the maximum in x_0 occurs near $\omega = \omega_0$. This is called resonance and is more pronounced, the smaller the value of γ. Note that at $\omega = 0$, $x_0 = (F_0/m\omega_0^2) = F_0/k$ which makes sense. The function then rises, peaks near ω_0 and vanishes as $\omega \to \infty$. See the book for some graphs, for different values of friction.

Once we have $x(t)$ we can take derivatives and get the answer of the velocity. The amplitude of velocity oscillations will not peak where x_0 does, though the two points will be close if γ is small.

Now for what is lacking in Eq. 16. Note it has no free parameters: both x_0 (the amplitude) and ϕ (the phase) are determined by m, γ, ω_0 and ω. How then do we arrange to have $x(0)$ and $v(0)$ equal to some arbitrary initial conditions? The answer is that to the $x(t)$ in Eqn.(16), which we will henceforth refer to as the *particular solution* $x_p(t)$ we can always add the complimentary function $x_c(t)$ from Eqn. (1) to get the answer

$$x(t) = x_p(t) + x_c(t) \tag{18}$$

$$= \frac{F_0}{m|I|}\cos(\omega t - \phi) \equiv x_0 \cos(\omega t - \phi) + Ce^{-\frac{\gamma t}{2}}\cos(\omega' t - \phi_0) \tag{19}$$

$$\omega' = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2} \tag{20}$$

Adding x_c will not affect the fact that Eqn (4) is satisfied since

$$\frac{d^2 x_c}{dt^2} + \gamma\frac{dx_c}{dt} + \omega_0^2 x_c = 0. \tag{21}$$

Thus $x = x_p + x_c$ obeys the requisite equation and has the two free parameters that allow us to choose our initial position and velocity at will.

Note however that due to the exponentially falling factor $e^{-\frac{\gamma t}{2}}$ in x_c, it will die down after some time. Thus x_c is called the transient solution and x_p, which goes on and on the steady-state solution. We will focus on the steady-state part from now on.

Let us admire some fine points. By using complex numbers we have managed to convert a differential equation Eq. (4) into an algebraic equation, Eq. (9). Next, note that the the response $x_p(t)$ is obtained from the cause $F_0 \cos\omega t$ by (i) dividing by $|I|$ and (ii) changing the phase by ϕ. This cannot be readily done in the world of real variables. However once problem is cast in terms of a complex force $F_0 e^{i\omega t}$ and its complex response, $z = z_0\ e^{i\omega t}$, the two are related by

$$z_0 = \frac{F_0}{mI(\omega)} \tag{22}$$

and division by a single complex number $I = |I|e^{i\phi}$ re-scales and shift the (amplitude of the) applied force to give the (amplitude of the) response.

For those of you who want the bottom line here it is. The driven oscillator has a complete solution given by

$$x(t) = x_p(t) + x_c(t) = \frac{F_0}{m|I|}\cos(\omega t - \phi) \equiv x_0 \cos(\omega t - \phi) + Ce^{-\frac{\gamma t}{2}}\cos(\omega' t - \phi_0)$$

$$\omega' = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$$

$$|I| = \sqrt{(-\omega^2 + \omega_0^2)^2 + (\omega^2\gamma^2)}$$

$$\tan\phi = \frac{\omega\gamma}{\omega_0^2 - \omega^2}$$

C and ϕ_0 are free parameters chosen to fit initial conditions