Problem Set V Solutions

1. *Consider masses m_1, m_2, m_3 at x_1, x_2, x_3. Find X, the CM coordinate by finding X_{12}, the CM of mass of 1 and 2, and combining it with m_3. Show this is gives the same result as*

$$X = \frac{\sum_{i=1}^{3} m_i x_i}{\sum_{i=1}^{3} m_i}.$$

The center of mass for masses 1 and 2 is given by

$$X_{12} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

The center of mass for all three masses is given by

$$\begin{aligned} X &= \frac{(m_1+m_2)X_{12} + m_3 x_3}{m_1+m_2+m_3} \\ &= \frac{(m_1+m_2)\frac{m_1x_1+m_2x_2}{m_1+m_2} + m_3x_3}{m_1+m_2+m_3} \\ &= \frac{m_1x_1+m_2x_2+m_3x_3}{m_1+m_2+m_3} \end{aligned}$$

as we would expect.

2. *Consider a square of mass 4 kg, side 2m, negligible thickness, with its sides oriented along the usual axes with its center at $(0,0)$. (i) Determine its CM using symmetry arguments. (ii) Imagine that the 1m $\times$ 1m part of it in the fourth quadrant is chopped off. Where is the new CM? Do this using the extension of result in previous problem. Repeat using the following trick: view the chopped off shape as the full square plus a 1m $\times$ 1m square of* negative *mass $-1kg$ in the fourth quadrant. (iii) A disk of radius R centered at the origin has a circular hole of radius $R/2$ centered at $(x = -R/2, y = 0)$. Where is its CM?*

   (i) The center of mass is located in the middle of the square. This must be the case because flipping the square around the x or y axis does not alter the square, so the center of mass cannot move during these rotations.

   (ii) Using the method from problem 1, we can consider this shape as three $1\,kg$ squares, one in each quadrent. Taking m_i to be the square in quadrant i, $\vec{X}_{12}$ will be $(x = 0, y = 0.5\,m)$. Then, the x coordinate for the center of mass will be

$$X = \frac{2\,kg(0) + 1\,kg(-0.5\,m)}{3\,kg} = -\frac{1}{6}m$$

and the y coordinate will be

$$Y = \frac{2\,kg(0.5\,m) + 1\,kg(-0.5\,m)}{3\,kg} = \frac{1}{6}m.$$

So, the center of mass is at $\vec{X} = (-1/6\,m, 1/6\,m)$.

For the second method, we can think of the square without the piece in the fourth quadrant as a mass of $M = 4kg$ at $(0,0)$ plus a mass of $-1kg = -M/4$ at (0.5 m, -0.5 m). The x coordinate of the center of mass is

$$\begin{aligned} x_{cm} &= \frac{(M)(0) + (-M/4)(0.5m)}{M - M/4} \\ &= \left(\frac{-1}{4}\right)\left(\frac{4}{3}\right)\left(\frac{1}{2}\right)m \\ &= \frac{-1}{6}m. \end{aligned}$$

Similarly, the y coordinate is

$$y_{cm} = \frac{(M)(0) + (-M/4)(-0.5m)}{M - M/4}$$
$$= \frac{1}{6}m.$$

As above, the center of mass of the square missing the piece in the fourth quadrant is $(-1/6\,m, 1/6\,m)$.

(iii) If the mass of the full disk is M, the mass of the missing piece will just be M times the ratio of the missing piece to the full piece.

$$M_{missing} = M\frac{\pi(R/2)^2}{\pi R^2} = \frac{M}{4}$$

Following the argument for the square, the center of mass of a circle must be in the middle of the circle. Therefore, we have a mass of M at (0,0) and a mass of $-M/4$ at (-R/2,0). Since both masses are centered at $y = 0$, y_{cm} must equal 0. Solving for x_{cm}

$$x_{cm} = \frac{M(0) + (-M/4)(-R/2)}{M - M/4} = \frac{R}{6}$$

The center of mass for the circle with a hole in it is at $(R/6, 0)$.

3. *Ideal Zorro (mass M, no height) swings down on a vine of length L from a height H and grabs a kid of mass m (zero height, standing on the ground) and together they barely reach safety at a height h. Relate H to the other parameters. Give H in meters if $L = 40m, M = 100kg, h = 6m, m = 30kg$.*

To solve this problem we first need to use conservation of energy, then conservation of momentum, and then conservation of energy again. Ideal Zorro starts at at heigh H. By conservation of energy, when he reaches be kid on the ground, his velocity v_1 will be

$$\frac{1}{2}Mv_1^2 = MgH \implies v_1 = \sqrt{2gH}.$$

On the ground, Zorro has an inelastic colision with the child, so momentum is conserved. Their combined final velocity v_2 just as they are leaving the ground is

$$Mv_1 = (M+m)v_2 \implies v_2 = \frac{M}{M+m}v_1. \tag{1}$$

Ideal Zorro and the child just reach a final heigh of h, so conserving energy,

$$\frac{1}{2}(M+m)v_2^2 = (M+m)gh \implies v_2 = \sqrt{2gh}. \tag{2}$$

Using equations 1 and 2 to solve for v_1,

$$v_1 = \frac{M+m}{M}\sqrt{2gh}$$

From above, we also have $v_1 = \sqrt{2gH}$. Setting these two equations equal and solving for H we find that

$$H = \left(\frac{M+m}{M}\right)^2 h = 10.1\,m.$$

4. *Two identical stars of mass M orbit around their CM. Show that*

$$T^2 = \frac{2\pi^2 R^3}{GM}$$

where R is the distance between planets. (Draw a figure and keep track of the gravitational force on either star as well as its centripetal acceleration.) (ii) Repeat for two unequal masses m and M and show

$$T^2 = \frac{4\pi^2 R^3}{G(M+m)}$$

where R is the separation between stars.

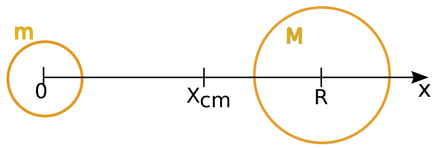

FIG. 1: Two stars of unequal mass orbiting about their center of mass.

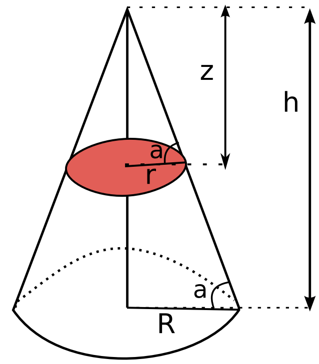

FIG. 2: A cone of heigh h and radius R. The angle a is the same no matter where you slice the cone, so this allows you to find an expression for r in terms of z.

(i) For two stars of equal mass M a distance R apart, the center of mass will be half way between the two stars. Therefore, the radius of their orbit will be $R/2$. To keep the stars in orbit, the gravitational acceleration must balance the centripetal acceleration. Also, remember that $v = 2\pi r f = 2\pi r/T$ where f is the frequency of rotation and T is the period of rotation.

$$F = \frac{GMM}{R^2} = \frac{Mv^2}{R/2}$$
$$\frac{GM}{R} = \frac{2(2\pi R/2)^2}{T^2} = \frac{2(\pi R)^2}{T^2}$$
$$T^2 = \frac{2\pi^2 R^3}{GM}$$

(ii) For two stars of unequal mass, their center of mass is no longer exactly between the two stars. If you take the star of mass m to be at $x = 0$ as shown in Figure 1, X_{cm} is given by

$$X_{cm} = \frac{MR}{m+M}.$$

The radius of star m's orbit will be given by $r = X_{cm}$, so balancing the gravitational force with the force necessary to maintain a circular orbit and solving for the period,

$$\frac{GMm}{R^2} = \frac{mv^2}{\frac{RM}{m+M}}$$
$$\frac{GM}{R^2} = \left(\frac{2\pi\frac{RM}{m+M}}{T}\right)^2 \frac{m+M}{RM}$$
$$\frac{G}{R^2} = \left(\frac{2\pi}{T}\right)^2 \frac{R}{m+M}$$
$$T^2 = \frac{4\pi^2 R^3}{G(m+M)}$$

as desired.

5. *Consider a massless boat of length L on frictionless water. At the left end is a person P_1 of mass m_1 holding a writhing snake of mass m_3 (Treat snake as rigid point particle. Treat P_1 who is clearly rigid at this point as a point.) At the right end is a person P_2 of mass m_2. At $t = 0$, P_1 throws the snake towards P_2 at a speed v. (i) What is V, the magnitude of the velocity of the boat and passengers relative to water when snake is airborne? (ii) How long does snake take to reach P_2? (iii) During this time how much has the boat moved to the left? (iv) Locate the CM at the end of all this snake throwing and show it is same as at beginning.*

The initial momentum of the snake, boat, person 1 and person 2 system is 0, so the total momentum of this system will remain 0 throughout the problem.

(i) The total momentum is zero, so

$$m_3 v - (m_1 + m_2)V = 0 \implies V = \frac{m_3}{m_1 + m_2} v.$$

The boat is moving to the left with a speed $V = \frac{m_3}{m_1+m_2} v$ relative to the water.

(ii) The time it takes the snake to reach P_2 will be the time in which it takes the snake and the boat to go a combined distance of L with the snake moving to the right and the boat moving to the left.

$$L = vt + Vt$$
$$L = \left(1 + \frac{m_3}{m_1 + m_2}\right) vt \implies t = \frac{L(m_1 + m_2)}{vM}$$

where $M = m_1 + m_2 + m_3$.

(iii) During this time, the boat has moved a distance

$$d = Vt = \frac{m_3}{m_1 + m_2} v \frac{L(m_1 + m_2)}{vM} \implies d = \frac{m_3}{M} L$$

or $\frac{m_3}{M} L$ to the left.

(iv) Taking P_1 to be at $x = 0$ before the snake was thrown, at the beginning of the problem

$$X_{cm} = \frac{m_2}{M} L.$$

The simplest way to find the center of mass after the snake is thrown is to again put P_1 at $x = 0$ and then shift the system by the amount found in (iii).

$$\begin{aligned} X_{cm} &= \frac{m_2 + m_3}{M} L - \frac{m_3}{M} L \\ &= \frac{m_2}{M} L \end{aligned}$$

Since there were no external forces acting on the snake, people, boat system, the center of mass has stayed the same throughout the problem.

6. *Find the CM of a cone of radius R and height h. (Think in terms of slices of thickness dy at height y.)*

We know by symmetry that the center of mass of a cone must lie along its axis which runs perpendicular to the base that goes through the point. If we consider a slice of the cone perpendicular to this line, it is a circle with radius r and width dz. Using Figure 2 it is easy to find the radius r in terms in the height z of the slice. We see that

$$\frac{h}{R} = \frac{z}{r} \implies r = R\frac{z}{h}.$$

The center of mass of each circle is along the axis of the cone, so we consider each circle as a point mass along the axis with mass $dm = \pi r^2 dz/(Volume\, of\, Cone)$. The volume of a cone is $\pi R^2 h/3$, so the center of mass is given by

$$\begin{aligned} Z_{cm} &= \int_0^h z dm \\ &= \int_0^h z \frac{\pi(R\frac{z}{h})^2}{\pi R^2 h/3} dz \\ &= \int_0^h 3z \frac{z^2}{h^3} dz \\ &= \frac{3}{4} h \end{aligned}$$

This says the center of mass is 3/4 of the way from the top to the base, or a distance $h/4$ from the base.

7. *A person of mass $M = 32.5kg$ on ice disdainfully throws my quantum text book weighing $m = 2.25kg$ at $v_b = 12m/s$. The book is thrown from zero height and the total distance between the book and the offender is $15.2m$ when the book lands. At what angle was this excellent book thrown? How fast is the offender moving?*

   Since there are no external forces in the horizontal direction on the person-book system, the x component of the center of mass does not move while the book is in the air. We can use this fact to find how far the book traveled before it hit the ground.

$$X_{cm} = 0 = x_b m + (x_b - x)M \implies x_b = x\frac{M}{m+M} \tag{3}$$

   where $x = 15.2\,m$ is the total final separation of the book and the person. Plugging in numbers, $x_b = 14.2\,m$.

   Take the book to be thrown at and angle α with respect to the horizontal. Then the book's initial velocity in the x direction is $v_b \cos\alpha$ and in the y direction is $v_b \sin\alpha$. In the x direction we know that

$$x_b = v_b t\cos\alpha \implies t = \frac{x_b}{v_b\cos\alpha}.$$

   In the y direction, the book starts and ends and the same height so,

$$0 = v_b t\sin\alpha - \frac{1}{2}gt^2 \implies v_b\sin\alpha = \frac{1}{2}gt.$$

   Plugging in for t, we find

$$\sin 2\alpha = \frac{gx_b}{2v_b^2} \implies \alpha = 37.6^0$$

   Taking the person's final velocity to be v, we know from conservation of momentum that

$$Mv = -mv_b\cos\alpha \implies v = -\frac{m}{M}v_b\cos\alpha.$$

   Plugging in, we find $v = -0.66m/s$. So, the book is thrown at an angle, $\alpha = 37.6^0$ and the offender has a final velocity of $-0.66\,m/s$. (Note: because of the double angle formula for α, there is a second solution when the angle is equal to $90 - \alpha$.)

8. *Block A of mass m is moving at velocity $+v$ towards mass B of mass $2m$ which is at rest. To its right and at rest is mass C of mass m. Find the ultimate velocities of all three masses assuming all collisions are elastic.*

   Since all the collisions are elastic, both energy and momentum are always conserved. First consider the collision of block A with block B. Conservation of energy and momentum tell us

$$\begin{aligned}\frac{1}{2}mv^2 = \frac{1}{2}mv_A^2 + \frac{1}{2}(2m)v_B'^2 &\implies v^2 = v_A^2 + 2v_B'^2\\ mv = mv_A + 2mv_B' &\implies v = v_A + 2v_B'\end{aligned}$$

   Solving these equations we find that $v_A = -v/3$ and $v_B' = 2v/3$.

   Setting up similar equations for the collision of B with C,

$$\begin{aligned}\frac{1}{2}(2m)v_B'^2 = \frac{1}{2}(2m)v_B^2 + \frac{1}{2}mv_C^2 &\implies 2v_B'^2 = 2v_B^2 + v_C^2\\ 2mv_B' = 2mv_B + mv_C &\implies 2v_B' = 2v_B + v_C\end{aligned}$$

   we find that $v_B = v_B'/3$ and $v_C = 4v_B'/3$. So, the final velocities for all the blocks are $v_A = -v/3$, $v_B = 2v/9$, $v_C = 8v/9$

9. *Two identical frictionless billiard balls are symmetrically hit by a third identical ball with velocity $\mathbf{i}v_0$. Find all subsequent velocities following this elastic collision. (Draw a picture at the moment of collision. What does "no friction" say about direction of forces?)*

   In this problem, both energy and momentum are conserved. However, I will start out by first by using symmetry to simplify the problem. The two balls that are hit (m_2 and m_3, see figure 9) have the same mass. Therefore, the magnitudes of their final velocities must be equal and they move in a direction $\pm\alpha$ relative to the horizontal.

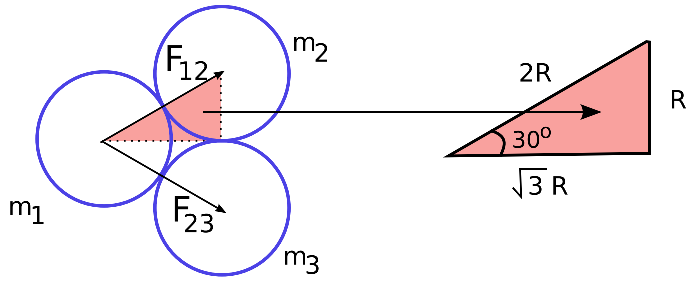

FIG. 3: Problem 9

From figure 9, it is easy to see that the angle α must be equal to 30^0. (The consequence of there being no friction is that the force between the billiard balls acts directly between the centers of mass. If there were friction acting at the contact point, the net force on the billiard ball would be in a different direction.)

With this information, we will now conserve energy and momentum. Conservation of energy plus the fact that $|\vec{v}_2| = |\vec{v}_3|$ tells us that

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 + \frac{1}{2}mv_3^2$$
$$v_0^2 = v_1^2 + 2v_2^2.$$

In the y direction, we know there is no initial momentum. From our symmetry arguments above, m_2 and m_3 will have equal and opposite components of momentum in the y direction. This sums to zero, so by conservation of momentum, m_1 will still have zero momentum in y following the collision. Conserving momentum in the x direction,

$$mv_0 = mv_1 + mv_2 \cos 30^0 + mv_3 \cos 30^0$$
$$v_0 = v_1 + 2v_2 \frac{\sqrt{3}}{2} \implies v_1^2 = v_0^2 + 3v_2^2 - 2\sqrt{3}v_0 v_2$$

Eliminating v_1^2 between the energy and momentum conservation equations we find,

$$v_2 = \frac{2\sqrt{3}}{5} v_0.$$

Plugging back into solve for v_1 and realizing that m_1 will move off in the minus x direction we find $v_1 = -v_0/5$. The final velocities following the collision are,

$$\vec{v}_1 = -v_0 \frac{1}{5}\,\hat{\imath}$$
$$\vec{v}_2 = v_0(\frac{2\sqrt{3}}{5}) \left(\frac{\sqrt{3}}{2}\,\hat{\imath} + \frac{1}{2}\,\hat{\jmath} \right)$$
$$\vec{v}_3 = v_0(\frac{2\sqrt{3}}{5}) \left(\frac{\sqrt{3}}{2}\,\hat{\imath} - \frac{1}{2}\,\hat{\jmath} \right).$$