Physics 200
Problem Set 7
Solution

Quick overview:
Although relativity can be a little bewildering, this problem set uses just a few ideas over and over again, namely

1. Coordinates (x, t) in one frame are related to coordinates (x', t') in another frame by the Lorentz transformation formulas.
2. Similarly, space and time intervals $(\Delta x, \Delta t)$ in one frame are related to intervals $(\Delta x', \Delta t')$ in another frame by the same Lorentz transformation formulas. Note that time dilation and length contraction are just special cases: it is time-dilation if $\Delta x = 0$ and length contraction if $\Delta t = 0$.
3. The spacetime interval $(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2$ between two events is the same in every frame.
4. Energy and momentum are always conserved, and we can make efficient use of this fact by writing them together in an energy-momentum vector $P = (E/c, \mathbf{p})$ with the property $P^2 = m^2c^2$. In particular, if the mass is zero then $P^2 = 0$.

1. The earth and sun are 8.3 light-minutes apart. Ignore their relative motion for this problem and assume they live in a single inertial frame, the Earth-Sun frame. Events A and B occur at $t = 0$ on the earth and at 2 minutes on the sun respectively. Find the time difference between the events according to an observer moving at $u = 0.8c$ from Earth to Sun. Repeat if observer is moving in the opposite direction at $u = 0.8c$.

   Answer: According to the formula for a Lorentz transformation,

$$\Delta t_{\text{observer}} = \gamma \left(\Delta t_{\text{Earth-Sun}} - \frac{u}{c^2} \Delta x_{\text{Earth-Sun}} \right), \qquad \gamma = \frac{1}{\sqrt{1 - (u/c)^2}}.$$

   Plugging in the numbers gives (notice that the c implicit in "light-minute" cancels the extra factor of c, which is why it's nice to measure distances in terms of the speed of light)

$$\Delta t_{\text{observer}} = \frac{2 \text{ min} - 0.8(8.3 \text{ min})}{\sqrt{1 - 0.8^2}} = -7.7 \text{ min},$$

   which means that according to the observer, event B happened *before* event A! If we reverse the sign of u then

$$\Delta t_{\text{observer 2}} = \frac{2 \text{ min} + 0.8(8.3 \text{ min})}{\sqrt{1 - 0.8^2}} = 14 \text{ min}.$$

2. Return to the Earth-Sun case above. (a) What is the speed of a spacecraft that makes the trip from the Sun to the Earth in 5 minutes according to the on board clocks? (b) What is the trip time in the Earth-Sun frame? (c) Find the square of the spacetime interval between them in light-minutes.

   (You may need to come back to part (c) after I do spacetime intervals in class. Do not just jump in and use some formula. Think in terms of events, assign as many possible spacetime coordinates as you can to each event in any frame and use the LT. Measure time in minutes, distance in light-minutes. Imagine a rod going from earth to the sun, if that helps.

Note that the spatial coordinate difference between events in the spacecraft frame are not the same as the distance between Earth and Sun in that frame. Even pre-Einstein, if I sit in my car going at 60 mph, I leave New Haven at $t = 0$ (Event 1) and arrive at Boston at $t = 2$ hrs (Event 2), the two events have the same coordinate in my frame (i.e., where I am in the car), $\Delta x = 0$, but that is not the distance between these towns.)

Answer:

(a) According to an observer on the spacecraft, $\Delta x_{\text{observer}} = 0$. So we can write (this is nothing but the time dilation formula, if you look carefully)

$$\Delta x_{\text{Earth-Sun}} = \gamma(0 + v\Delta t_{\text{observer}}) \implies \frac{\Delta x_{\text{Earth-Sun}}}{\Delta t_{\text{observer}}} = v\gamma = \frac{v}{\sqrt{1-(v/c)^2}} = \frac{c}{\sqrt{(c/v)^2 - 1}},$$

and solving for v gives

$$v = c\left[1 + \left(\frac{\Delta t_{\text{observer}}}{\Delta x_{\text{Earth-Sun}}/c}\right)^2\right]^{-1/2} = (3\times 10^8 \text{ m/s})\left[1 + \left(\frac{5 \text{ min}}{8.3 \text{ min}}\right)^2\right]^{-1/2} = 2.6\times 10^8 \text{ m/s}.$$

(b) In the Earth-Sun frame

$$\Delta t_{\text{Earth-Sun}} = \frac{\Delta x_{\text{Earth-Sun}}}{v} = \frac{8.3 \text{ light-minutes}}{2.6\times 10^8 \text{ m/s}} = 9.6 \text{ min}.$$

Alternatively, we can use the time-dilation formula to get (the difference is due to rounding-errors)

$$\Delta t_{\text{Earth-Sun}} = \frac{\Delta t_{\text{observer}}}{\sqrt{1-(v/c)^2}} = \frac{5 \text{ min}}{\sqrt{1 - \left[1 + \left(\frac{5 \text{ min}}{8.3 \text{ min}}\right)^2\right]^{-1}}} = 9.7 \text{ min}.$$

Note that we can use the "regular" time-dilation formula because, in effect, there is only one clock on the spacecraft. Before, we had to worry about two different times at two different locations, which is why we had to use the full Lorentz transformation.

(c) Since the spacetime interval is the same in every frame, we can choose a frame where it can be most easily evaluated, for example in the frame where $\Delta x = 0$. This is true on board the spacecraft, and we already know what the time on board is, so

$$(\Delta s)^2 = (c\Delta t_{\text{observer}})^2 = (c\cdot 5 \text{ min})^2 = 25 \text{ (light-minutes)}^2 = 9\times 10^4 \text{ (light-seconds)}^2$$

Let's check that our calculations are correct by doing the same in the Earth-Sun frame:

$$(\Delta s)^2 = (c\Delta t_{\text{Earth-Sun}})^2 - (\Delta x_{\text{Earth-sun}})^2 = (c\cdot 9.7 \text{ min})^2 - (8.3 \text{ light-minutes})^2 = 25 \text{ (light-minutes)}^2,$$

which agrees with the previous answer, as expected.

3. A muon has a lifetime of 2×10^{-6} s in its rest frame. It is created 100 km above the earth and moves towards it at a speed of 2.97×10^8 m/s. At what altitude does it decay? According to the muon, how far did it travel in its brief life?

Answer: The time dilation factor is $\gamma = 1/\sqrt{1-(v/c)^2}$, and the muon's lifetime τ in its rest frame corresponds to $\gamma\tau$ in the laboratory frame. This means the muon travels a distance $v\gamma\tau$, or

$$d = v\gamma\tau = \frac{(2.97\times 10^8 \text{ m/s})(2\times 10^{-6} \text{ s})}{\sqrt{1 - \left(\frac{2.97\times 10^8 \text{ m/s}}{3\times 10^8 \text{ m/s}}\right)^2}} = 4.2 \text{ km}$$

before it decays, which occurs at 95.8 km above the ground. According to the muon, it has only traveled

$$\frac{d}{\gamma} = v\tau = (2.97 \times 10^8 \text{ m/s})(2 \times 10^{-6} \text{ s}) = 590 \text{ m}.$$

4. An observer S who lives on the x-axis sees a flash of red light at $x = 1210$ m, then after 4.96 μs, a flash of blue at $x = 480$ m. Use subscripts R and B to label the coordinates of the events.

   (i) What is the velocity relative to S of an observer S' who records the events as occurring at the same place?

   (ii) Which event occurs first according to S' and what is the measured time interval between these flashes? For the former you do not need to do a calculation. For the latter I suggest using the spacetime interval.

   Answer:

   (i) Physically, what happens is that the observer sees the flash of red, then travels at velocity v such that he arrives just in time to see the flash of blue. So without thinking too hard we can say

   $$x_R - vt_R = x_B - vt_B,$$

   or

   $$v = \frac{x_R - x_B}{t_R - t_B} = \frac{1210 \text{ m} - 480 \text{ m}}{0 - 4.96 \times 10^{-6} \text{ s}} = -1.47 \times 10^8 \text{ m/s}.$$

   (ii) We have found in part (i) an observer who can travel from one event to the other at $v < c$, and since the spacecraft is as good a signal as any, from what we learned in class the order of the events must be the same in every frame. Therefore the observer in S' sees event R happen before event B. To find the time interval in S', we can use the invariant spacetime interval and use the fact that the events occur at the same place in S':

   $$(c\Delta t')^2 - (\Delta x')^2 = (c\Delta t')^2 = (c\Delta t)^2 - (\Delta x)^2.$$

   Therefore

   $$\Delta t' = \sqrt{(\Delta t)^2 - \left(\frac{\Delta x}{c}\right)^2} = \sqrt{(4.96 \times 10^{-6} \text{ s})^2 - \left(\frac{1210 \text{ m} - 480 \text{ m}}{3 \times 10^8 \text{ m/s}}\right)^2} = 4.32\ \mu\text{s}.$$

   In case you are suspicious, we can do this explicitly, using the value of v found above:

   $$t'_R = \gamma\left(t_R - \frac{v}{c^2}x_R\right) = \frac{1}{\sqrt{1 - \left(\frac{1.47\times 10^8 \text{ m/s}}{3\times 10^8 \text{ m/s}}\right)^2}}\left[0 - \frac{-1.47 \times 10^8 \text{ m/s}}{(3 \times 10^8 \text{ m/s})^2}(1210 \text{ m})\right] = 2.27\ \mu\text{s},$$

   $$t'_B = \gamma\left(t_R - \frac{v}{c^2}x_R\right) = \frac{1}{\sqrt{1 - \left(\frac{1.47\times 10^8 \text{ m/s}}{3\times 10^8 \text{ m/s}}\right)^2}}\left[4.96 \times 10^{-6} \text{ s} - \frac{-1.47 \times 10^8 \text{ m/s}}{(3 \times 10^8 \text{ m/s})^2}(480 \text{ m})\right] = 6.59\ \mu\text{s}.$$

   Again, event R occurs before event B, and the time interval between these flashes is $\Delta t' = t'_B - t'_R = 4.32\ \mu$s.

5. Two rockets of rest length L_0 are approaching the earth from opposite directions at velocities $\pm c/2$. How long does one of them appear to the other?

   Answer: Let's pick one rocket (call it rocket 1) and consider how fast the other rocket (rocket 2) looks in this frame. In the Earth frame, rocket 1 has velocity $c/2$ and rocket 2 has velocity $-c/2$. Applying the velocity addition law gives

$$v_2' = \frac{v_2 - v_1}{1 - v_1 v_2/c^2} = \frac{(-c/2) - (c/2)}{1 - (c/2)(-c/2)/c^2} = -\frac{4}{5}c,$$

   i.e., rocket 2 looks like it is approaching at $\frac{4}{5}c$. Applying the Lorentz contraction formula gives

$$L' = \frac{L_0}{\gamma} = L_0\sqrt{1 - \left(\frac{4}{5}\right)^2} = \frac{3}{5}L_0.$$

6. A body quadruples its momentum when its speed doubles. What was the initial speed in units of c, i.e., what was u/c?

   Answer: With f standing for final, i for initial, and $v_f = 2v_i$, we have

$$\frac{p_f}{p_i} = \frac{\gamma_f m v_f}{\gamma_i m v_i} = \frac{2\gamma_f}{\gamma_i} = 4 \implies \frac{1 - (v_i/c)^2}{1 - (2v_i/c)^2} = 4,$$

   after writing out the γ's and squaring both sides. Solving for v_i/c gives

$$\frac{v_i}{c} = \frac{1}{\sqrt{5}}.$$

7. A body of rest mass m_0 moving at speed v collides with and sticks to an identical body at rest. What is the mass and momentum of the final clump?

   Answer: The energy-momentum vector of the moving mass is $P_1 = (\gamma m_0 c, \gamma m_0 v)$ where $\gamma = 1/\sqrt{1 - (v/c)^2}$, while for the mass at rest it is $P_2 = (m_0 c, 0)$. Thus the total energy-momentum vector of the final clump is $P = P_1 + P_2$. We can read off the momentum as $\gamma m_0 v$, and for the mass

$$P^2 = M^2c^2 = (P_1 + P_2)^2 = P_1^2 + 2P_1 \cdot P_2 + P_2^2 = m_0^2c^2 + 2\gamma m_0^2 c^2 + m_0^2 c^2 = 2m_0^2c^2(1+\gamma),$$

   so that

$$M = m_0\sqrt{2(1+\gamma)}.$$

8. A body of rest mass m_0 moving at speed v approaches an identical body at rest. Find V, the speed of a frame in which the total momentum is zero. Do this first by the law of composition of velocities starting with how you would do this non-relativistically. Next, repeat using the transformation law for the components of the energy-momentum vector. (You may need to come back to the second part after I do energy-momentum vectors in class.)

   Answer (see also the solution to Problem 11): First we use velocity addition. You should convince yourself that if two identical particles have equal and opposite velocities they have equal and opposite momenta, in both the classical case where $p = mv$ and in the relativistic case where $p = \gamma m v$. So we will look for a frame where the velocities are equal and opposite. In a frame moving at velocity V with respect to the original frame, the body at rest has velocity $-V$, while the mass that was already moving at v has velocity

$$v' = \frac{v - V}{1 - vV/c^2}$$

with respect to the new frame. If the total momentum is zero, then

$$\frac{v-V}{1-vV/c^2}=V \implies \tilde{v}\tilde{V}^2-2\tilde{V}+\tilde{v}=0,$$

where we have let $\tilde{v}=v/c$ and $\tilde{V}=V/c$ (basically, set $c=1$ temporarily) in order not to make the solution look more complicated than it is. Note that this is the relativistic generalization of the equation $v-V=V$ for equal masses. In any case, the quadratic equation gives

$$\tilde{V}=\frac{1}{\tilde{v}}\left(1\pm\sqrt{1-\tilde{v}^2}\right).$$

It's clear that the solution with a plus sign makes $\tilde{V}$ larger than 1 (never true) for $\tilde{v}<1$ (always true), so we must take the solution with a minus sign. Moreover, we can actually rewrite this in a nice way if we let

$$\gamma=\frac{1}{\sqrt{1-(v/c)^2}}=\frac{1}{\sqrt{1-\tilde{v}^2}},$$

so that

$$\tilde{V}=\tilde{v}\frac{1-\frac{1}{\gamma}}{\tilde{v}^2}=\tilde{v}\frac{1-\frac{1}{\gamma}}{1-\frac{1}{\gamma^2}}=\tilde{v}\frac{1-\frac{1}{\gamma}}{\left(1-\frac{1}{\gamma}\right)\left(1+\frac{1}{\gamma}\right)}=\tilde{v}\frac{1}{1+\frac{1}{\gamma}}=\tilde{v}\frac{\gamma}{1+\gamma},$$

or

$$V=v\frac{\gamma}{1+\gamma}.$$

Next, we use the energy-momentum vector. In the original frame, the energy-momentum vector of the moving mass is $(\gamma m_0c, \gamma m_0 v)$ where $\gamma=1/\sqrt{1-(v/c)^2}$, while for the mass at rest it is $(m_0c, 0)$. Thus the total energy-momentum vector is $P=(m_0c(1+\gamma), \gamma m_0 v)$. In the frame with zero total momentum we must have $P'=(E/c, 0)$, so if we write out the transformation law for the momentum part of this vector we have

$$0=\frac{\gamma m_0 v-\frac{V}{c}[m_0c(1+\gamma)]}{\sqrt{1-(V/c)^2}} \implies \gamma m_0 v=m_0V(1+\gamma).$$

Solving for V is a simple matter, giving

$$V=v\frac{\gamma}{1+\gamma},$$

which agrees with our answer above. The second method is clearly the better method, however.

At this point it is also good to check that our answer is reasonable in the non-relativistic limit, i.e., when $\gamma\simeq 1$. Then $V=v/2$, which is what we would have obtained without relativity.

9. Show that a photon cannot break up into an electron and a positron. For our purposes the electron and positron are identical particles with four-momenta (with $c=1$) $P_1=m_0(\gamma_1,\ \gamma_1\mathbf{v}_1)$ and $P_2=m_0(\gamma_2,\ \gamma_2\mathbf{v}_2)$ where $\gamma=1/\sqrt{1-v^2}$. The photon four-momentum is $K=(\omega,\mathbf{k})$ with $\omega=|\mathbf{k}|$. I suggest you use energy-momentum conservation and the dot products of four vectors to show that this process is kinematically forbidden.

   Answer: The conservation of energy-momentum tells us $K=P_1+P_2$, so if we square both sides we get

$$K^2=0=(P_1+P_2)^2=P_1^2+2P_1\cdot P_2+P_2^2=m_0^2c^2+2\left(\frac{E_1E_2}{c^2}-\mathbf{p}_1\cdot\mathbf{p}_2\right)+m_0^2c^2,$$

which would imply

$$\mathbf{p}_1 \cdot \mathbf{p}_2 = m_0^2 c^2 + \frac{E_1}{c}\frac{E_2}{c}.$$

But this is impossible, because the formula

$$\left(\frac{E}{c}\right)^2 = p^2 + m^2 c^2$$

which holds for all particles tells us $p_1 < E_1/c$ and $p_2 < E_2/c$ for particles with mass. Therefore this process is forbidden by relativity.

10. **In this problem let** $c = 1$. A particle of rest mass m_0 decays at rest into a photon and loses rest mass δ in the bargain. Show that the photon energy is $\nu = \delta(1 - \frac{\delta}{2m_0})$ in the particle's rest frame before the decay. Note that the photon has zero rest mass, that is, the square of its four momentum vanishes. The problem can be solved most efficiently if you use four vectors and dot products between them, but I do not insist you do. Denote the four vector of the photon by $K = (\nu, \mathbf{k})$ and that of the particle before and after by $P = (E, \mathbf{p})$ and $P' = (E', \mathbf{p}')$.

    Answer: With the notation given in the problem, the conservation of energy-momentum for the decay is $P = K + P'$ or $P' = P - K$. Using the fact that $P = (m_0, \mathbf{0})$ (the particle is initially at rest), we square both sides to get

    $$P'^2 = (m_0 - \delta)^2 = (P - K)^2 = P^2 - 2P \cdot K + K^2 = m_0^2 - 2P \cdot K = m_0^2 - 2m_0\nu.$$

    Expanding $(m_0 - \delta)^2$ and canceling the m_0^2's gives

    $$-2m_0\delta + \delta^2 = -2m_0\nu \implies \nu = \delta\left(1 - \frac{\delta}{2m_0}\right).$$

11. Consider a particle of rest mass m_0 moving at velocity v in your frame S. Write down expressions for the components of its energy momentum vector $P = (p_0, p_1)$ in terms of m_0, v. Now see this particle from a frame S' moving at velocity u. What will be its velocity w and what will be the components of $P' = (p_0', p_1')$, first in terms of w and then in terms of w written in terms of u and v? Show that the primed coordinates are related to unprimed ones by the same Lorentz Transformation that relates (x_0, x_1) to (x_0', x_1').

    Answer:

    $$P = (\gamma_v m_0 c, \gamma_v m_0 v), \qquad \gamma_v = \frac{1}{\sqrt{1 - (v/c)^2}}.$$

    From the formula for the addition of velocities, in S' the particle has velocity

    $$w = \frac{v - u}{1 - vu/c^2},$$

    so that

    $$P' = (\gamma_w m_0 c, \gamma_w m_0 w), \qquad \gamma_w = \frac{1}{\sqrt{1 - (w/c)^2}}$$

    (the meaning of the labels on γ should be obvious by now). The expression quickly gets messy if we just try to write this in terms of u, v, so we do some algebra first:

    $$\gamma_w = \frac{1}{\sqrt{1 - \frac{1}{c^2}\left(\frac{v-u}{1-vu/c^2}\right)^2}} = \frac{c^2 - vu}{\sqrt{(c^2 - vu)^2 - c^2(v-u)^2}} = \frac{c^2 - vu}{\sqrt{c^4 + v^2u^2 - c^2v^2 - c^2u^2}}$$

    $$= \frac{c^2 - vu}{\sqrt{(c^2 - u^2)(c^2 - v^2)}} = \frac{1 - vu/c^2}{\sqrt{1 - (u/c)^2}\sqrt{1 - (v/c)^2}} = \gamma_u\gamma_v\left(1 - \frac{vu}{c^2}\right).$$

Then

$$p'_0 = \gamma_w m_0 c = \gamma_u \left(\gamma_v m_0 c - \frac{u}{c^2}\gamma_v m_0 v\right) = \gamma_u \left(p_0 - \frac{u}{c^2} p_1\right),$$
$$p'_1 = \gamma_w m_0 w = \gamma_u \gamma_v m_0 \left(1 - \frac{vu}{c^2}\right) \frac{v-u}{1 - vu/c^2} = \gamma_u \left(\gamma_v m_0 v - \frac{u}{c}\gamma_v m_0 c\right) = \gamma_u \left(p_1 - \frac{u}{c} p_0\right),$$

which is precisely how (x_0, x_1) transforms into (x'_0, x'_1).

12. (OPTIONAL) Consider two rockets A and B of rest length $L_0 = 1$ m travelling towards each other with a tiny shift in the y-direction so they do not collide, as in part Fig. 1. Each sees the other approach it with speed u. According to A, when the tail of B passed the tip of A, a missile was fired from the tail of A towards B as in part (a) of the figure. It will clearly miss due to length contraction of B as seen by A. But B will see the event as in part (b) of the figure and expect a hit. Who is right? (Ignore the time it takes the missile to hit its target.)

It will be most educational to assign spacetime coordinates to five events in each frame: tip of A passes tip of B (set it to $(0, 0)$ for both), tail of B passes tip of A, missile is fired, tip of B passes tail of A and finally tails pass. It will be useful to know that if an event occurs at either end of either rockets its spatial coordinates are no-brainers *in that rocket frame* since the tips are always at $x = 0, x' = 0$ and the tails are at $x = -1$ and $x' = +1$. It should also be easy to find time elapsed between tip of your rocket passing my tip and my tail since I am of unit length and you are moving at speed u. Ditto for tail. As a check of your coordinates you may want to see that the spacetime interval between any event and $(0, 0)$ comes out same for both.

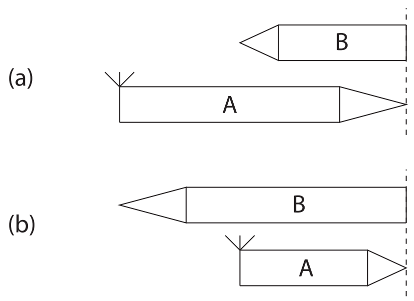

Figure 1: The point of view of A is in (a). We think B will see it as in (b). But something is wrong since there was no hit according to A. The three funny lines coming out of the tip of A are supposed to be the missile aimed at B.

Answer: Let $\gamma = 1/\sqrt{1 - (u/c)^2}$. It's easy to see that the sequence of events is simpler in frame A, so keeping in mind that according to A the other rocket has length L_0/γ and velocity $-u$ we begin by writing down the coordinates of the following events:

$$\begin{aligned}
\text{Tip of } B \text{ passes tip of } A\text{:}\ & (x, t) = (0, 0) \\
\text{Tail of } B \text{ passes tip of } A\text{:}\ & (0, L_0/u\gamma) \\
\text{(M) Missile is fired from tail of } A\text{:}\ & (-L_0, L_0/u\gamma) \\
\text{(T) Tip of } B \text{ passes tail of } A\text{:}\ & (-L_0, L_0/u) \\
\text{Tail of } B \text{ passes tail of } A\text{:}\ & (-L_0, L_0/u + L_0/u\gamma).
\end{aligned}$$

Note that because γ is greater than 1, the missile is fired (M) before the tip of B reaches the tail of A (T), and the missile *misses* rocket B according to A. Knowing that relativity is a consistent theory of physics, we expect B to conclude the same, but it's not immediately obvious that this is the case. So let's use the Lorentz transformation formulas (recall that B moves at velocity $-u$ relative to A)

$$x_B = \gamma(x_A + ut_A),$$
$$t_B = \gamma\left(t_A + \frac{u}{c^2}\,x_A\right)$$

to see what transpires according to B:

$$\begin{aligned}
\text{Tip of } A \text{ passes tip of } B\text{:}\ & (0,0)\\
\text{Tip of } A \text{ passes tail of } B\text{:}\ & (L_0, L_0/u)\\
\text{(M) Missile is fired from tail of } A\text{:}\ & (L_0 - \gamma L_0, L_0/u - \gamma u L_0/c^2)\\
\text{(T) Tail of } A \text{ passes tip of } B\text{:}\ & (0, L_0/u\gamma)\\
\text{Tail of } A \text{ passes tail of } B\text{:}\ & (L_0, L_0/u + L_0/u\gamma).
\end{aligned}$$

These are the same events as above described from the point of view of B. We can now compare the times when B sees the missile being fired (M) and the tail of A passing the tip of B (T):

$$t_T = \frac{L_0}{u\gamma}, \qquad t_M = \frac{L_0}{u\gamma}\left(\gamma - \frac{\gamma^2 u^2}{c^2}\right).$$

It turns out that the quantity in parentheses is always less than 1 if u is not zero and is less than the speed of light, which we can see by letting $z = u/c$ and writing it as

$$\frac{1}{\sqrt{1-z^2}} - \frac{z^2}{1-z^2} = \frac{\sqrt{1-z^2} - z^2}{1-z^2}.$$

The square root in the numerator is always less than 1, so the numerator is always less than the denominator, i.e., the quantity in parentheses is always less than 1. So we've shown that t_M is always less than t_T, which in turn shows that according to B, the missile fires (M) *before* the tail of A passes the tip of B (T). Therefore the missile also misses according to B. The point is that in B's frame the tip of A passing the tail of B is not simultaneous with the firing of the missile, and in fact occurs *after* the missile is fired. Strange, but true.

Since both observers agree that the missile misses the rocket, in fact both are right: they agree that the missile misses.