

## Problem Set XI Solutions

### Fall 2006 Physics 200a

1. How much heat is needed to convert 1 kg of ice at  $-10^{\circ}\text{C}$  to steam at  $100^{\circ}\text{C}$ ? Remember ice and water do not have the same specific heat.

In general the heat necessary to warm a material that doesn't change phase is:

$$Q = \Delta T \cdot c \cdot m$$

Where here the temperature change is in Kelvin, the specific heat  $c$  is in  $\text{J}/\text{kg}\cdot\text{K}$ , and the mass is in  $\text{kg}$ . Also in general, the energy required to change the phase of a mass  $m$  with heat of transformation  $L$  is:

$$Q = L \cdot m$$

The heat necessary to warm the ice to  $0^{\circ}\text{C}$  is:

$$Q_{ice} = (10\text{K}) \left( 2050 \frac{\text{J}}{\text{kg}\cdot\text{K}} \right) (1\text{kg}) = 20.5 \text{ kJ}$$

We can look up the heat of fusion of water to get:

$$Q_{ice \rightarrow water} = \left( 334 \frac{\text{kJ}}{\text{kg}} \right) (1\text{kg}) = 334 \text{ kJ}$$

The heat then necessary to warm the water from  $0^{\circ}\text{C}$  to  $100^{\circ}\text{C}$  is:

$$Q_{water} = (100\text{K}) \left( 4184 \frac{\text{J}}{\text{kg}\cdot\text{K}} \right) (1\text{kg}) = 418.4 \text{ kJ}$$

And finally, we look up the heat of vaporization of water getting:

$$Q_{water \rightarrow steam} = \left( 2257 \frac{\text{kJ}}{\text{kg}} \right) (1\text{kg}) = 2257 \text{ kJ}$$

So, the total heat needed is:

$$Q_{total} = Q_{ice} + Q_{ice \rightarrow water} + Q_{water} + Q_{water \rightarrow steam}$$
$$Q_{total} = 20.5 \text{ kJ} + 334 \text{ kJ} + 418.4 \text{ kJ} + 2257 \text{ kJ} \approx 3030 \text{ kJ}$$

2. If 400g of ice at  $-2^{\circ}\text{C}$  is placed in 1 kg of water at  $21^{\circ}\text{C}$  what is the end product when equilibrium is reached?

First we need to determine if the ice will completely melt. To do this we find the heat necessary to warm the ice to  $0^{\circ}\text{C}$  and then to melt it with the same method as in Prob. 1.

$$Q_{ice} = (2\text{K}) \left( 2050 \frac{\text{J}}{\text{kg}\cdot\text{K}} \right) (.4\text{kg}) = 1.64 \text{ kJ}$$

$$Q_{ice \rightarrow water} = \left( 334 \frac{\text{kJ}}{\text{kg}} \right) (.4\text{kg}) = 133.6 \text{ kJ}$$

$$Q_{melt} = Q_{ice} + Q_{ice \rightarrow water} = 135.24 \text{ kJ}$$

The maximum amount of heat that the water can give before beginning to freeze is:

$$Q_{water} = (21K) \left( 4184 \frac{J}{kg \cdot K} \right) (1kg) = 87.864 kJ$$

Since this is smaller than the amount of heat necessary to completely melt but far more than the amount necessary to warm the ice to  $0^\circ C$  we now know that the water will warm all of the ice to  $0^\circ C$  and melt some fraction of the ice. We already know that  $1.64 kJ$  is necessary to warm the ice to  $0^\circ C$ , so the remaining  $86.224 kJ$  available from the water will go toward melting the ice. The amount of ice melted is then:

$$m = \frac{Q}{L_f} = \frac{86.224 kJ}{334 kJ/kg} \approx 258 g$$

Thus we're left with about  $1.258 kg$  of water and  $0.142 kg$  of ice, all at  $0^\circ C$ .

3. To find  $c_x$ , the specific heat of material  $X$ , I place  $75g$  of it in a  $30g$  copper calorimeter that contains  $65g$  of water, all initially at  $20^\circ C$ . When I add  $100g$  of water at  $80^\circ C$ , the final temperature is  $49^\circ C$ . What is  $c_x$ ?

Here we want to make use of conservation of energy, namely the heat absorbed by the original amount of material  $X$ , the calorimeter and the initial water is equal to the heat lost by the added hot water. This gives us the equation:

$$(m_x c_x + m_{copper} c_{copper} + m_{water} c_{water}) \Delta t_{initial.components} = m_{hot.water} c_{hot.water} \Delta t_{hot.water}$$

where both temperature changes are positive numbers.

$$c_x = \frac{1}{m_x} \left( m_{hot.water} c_{hot.water} \frac{\Delta t_{hot.water}}{\Delta t_{initial.components}} - m_{copper} c_{copper} - m_{water} c_{water} \right)$$

If we look up values for the specific heats of water and copper, we can now solve this equation.

$$c_x = \frac{1}{.075 kg} \left[ (.1 kg) \left( 4184 \frac{J}{kg \cdot K} \right) \left( \frac{31K}{29K} \right) - (.03 kg) \left( 386 \frac{J}{kg \cdot K} \right) - (.065 kg) \left( 4184 \frac{J}{kg \cdot K} \right) \right]$$

$$c_x \approx 2180 \frac{J}{kg \cdot K}$$

4. How many moles of ideal gas are there in a room of volume  $50m^3$  at atmospheric pressure and  $300K$ ?

The ideal gas law tells us that  $PV = nRT$ . Noting that Earth's normal atmospheric pressure is  $1.013 \cdot 10^5 Pa$  and solving for the number of moles,  $n$  gives:

$$n = \frac{PV}{RT} = \frac{(1.013 \cdot 10^5 Pa)(50m^3)}{\left( 8.314 \frac{J}{K \cdot mol} \right) (300K)} \approx 2030 mol$$

5. A spherical air bubble of radius  $2\text{cm}$  is released  $30\text{m}$  below the surface of a pond at  $280\text{K}$ . What is its volume when it reaches the surface, which is at  $300\text{K}$  assuming it is in thermal equilibrium the whole time? Ignore the size of the bubble compared to other dimensions like  $30\text{m}$ .

Using the ideal gas law and the fact that neither the universal gas constant  $R$  nor the number of moles of air are changing we can write the relation:

$$\frac{P_i V_i}{T_i} = nR = \frac{P_f V_f}{T_f}$$

Which is easily solved for the final volume:

$$V_f = V_i \left( \frac{P_i}{P_f} \right) \left( \frac{T_f}{T_i} \right)$$

The initial volume is easily found since we know that the volume of a sphere is:

$$V_{\text{sphere}} = \frac{4}{3} \pi r^3$$

Since we can ignore the size of the bubble compared to the other length scales in the problem, the final pressure is just the atmospheric pressure, and the initial pressure is the atmospheric pressure plus the pressure from the water. Generalizing for a fluid density  $\rho$  and depth  $h$ :

$$P_f = P_{\text{atm}}$$

$$P_i = P_{\text{atm}} + \rho g h$$

Since the initial and final temperatures were specified in the problem we can now get a solution.

$$V_f = \frac{4\pi}{3} (0.02\text{m})^3 \left[ \frac{1.013 \cdot 10^5 \text{Pa} + (1000 \text{kg/m}^3)(9.8 \text{m/s}^2)(30\text{m})}{1.013 \cdot 10^5 \text{Pa}} \right] \left( \frac{300\text{K}}{280\text{K}} \right)$$

$$V_f \approx 1.4 \cdot 10^{-4} \text{m}^3$$

6. What is the volume of one mole of an ideal gas at STP: Standard Temperature ( $273\text{K}$ ) and Pressure (1 atmosphere)?

The ideal gas law tells us that  $PV = nRT$ . We also know that 1 atmosphere =  $1.013 \cdot 10^5 \text{Pa}$  and that  $R = 8.314 \text{J/mol}\cdot\text{K}$ . So:

$$V = \frac{nRT}{P} = \frac{(1\text{mol}) \left( 8.314 \frac{\text{J}}{\text{mol}\cdot\text{K}} \right) (273\text{K})}{1.013 \cdot 10^5 \text{Pa}} \approx 0.0224 \text{m}^3$$

7. One mole of ideal Nitrogen gas is at 2 atmospheres and occupies a volume of  $10\text{m}^3$ . Find  $T$  in Kelvin,  $U$  the internal energy (assumed to be just kinetic energy) in Joules, and the typical velocity of the gas molecules which have a mass  $4.65 \cdot 10^{-26} \text{kg}$ ?

We can find the temperature in Kelvin after simple manipulation of the ideal gas law.

$$T = \frac{PV}{nR} = \frac{(2\text{atm})(1.013 \cdot 10^5 \text{ Pa/atm})(10\text{m}^3)}{(1\text{mol})\left(8.314 \frac{\text{J}}{\text{mol} \cdot \text{K}}\right)} \approx 244000 \text{ K}$$

(Although this is clearly a very high temperature, the idea gas law is still valid since the density is low.)

If we assume that the internal energy is just kinetic energy then kinetic theory tells us:

$$U = \frac{1}{2} m \overline{v^2} = \frac{3}{2} kT$$

where  $k = 1.38 \cdot 10^{-23} \text{ J/K}$  is Boltzmann's constant. Thus:

$$U = \frac{3}{2} \frac{kPV}{nR} = \frac{3}{2} \frac{(1.38 \cdot 10^{-23} \text{ J/K})(2\text{atm})(1.013 \cdot 10^5 \text{ Pa/atm})(10\text{m}^3)}{(1\text{mol})\left(8.314 \frac{\text{J}}{\text{mol} \cdot \text{K}}\right)} \approx 50.4 \cdot 10^{-19} \text{ J}$$

$$v_{\text{typical}} = \sqrt{\frac{2U}{m}} \approx \sqrt{\frac{2(50.4 \cdot 10^{-19} \text{ J})}{4.65 \cdot 10^{-26} \text{ kg}}} \approx 14700 \text{ m/s}$$

- 8. A copper rod of length 50cm and radius 2cm has one end dipped in an ice-water mixture and the other in boiling water. What is the heat flow  $dQ/dt$ ?**

The heat flow through a material with thermal conductivity  $k$ , thickness  $\Delta x$ , area  $A$ , and a temperature difference of  $\Delta T$  is:

$$\frac{dQ}{dt} = -kA \frac{\Delta T}{\Delta x}$$

where the minus sign ensures that heat flows from the hot side to the cold side. Since the area of a circle is  $A = \pi r^2$ , the temperature difference between boiling water and ice water is 100K, and the thermal conductivity of copper is 401 W/m·K, here we have:

$$\frac{dQ}{dt} = -\pi \left(401 \frac{\text{W}}{\text{m} \cdot \text{K}}\right) (0.02\text{m})^2 \left(\frac{100\text{K}}{0.50\text{m}}\right) \approx -101 \text{ W}$$

- 9. How much heat flows out per second through a concrete roof of area 100m<sup>2</sup> and thickness 20cm if the outside is at 0°C and the inside is at 17°C.**

The thermal conductivity for concrete is about 1 W/m·K and as in Prob. 8:

$$\frac{dQ}{dt} = -kA \frac{\Delta T}{\Delta x}$$

So here the heat flowing out of the roof is:

$$\frac{dQ}{dt} = -\left(1 \frac{\text{W}}{\text{m} \cdot \text{K}}\right) (100\text{m}^2) \left(\frac{-17\text{K}}{0.20\text{m}}\right) = 8500 \text{ W}$$

10. A gas goes over the cycle ABCA as in Figure 1 where AC is an isotherm and AB is an isobar. (Note  $L$  stands for Liter, with  $L=10^{-3}m^3$ .) Find the  $(P, V)$  coordinates of C. What is the work done in each part of the cycle and the heat absorbed or rejected in the full cycle?

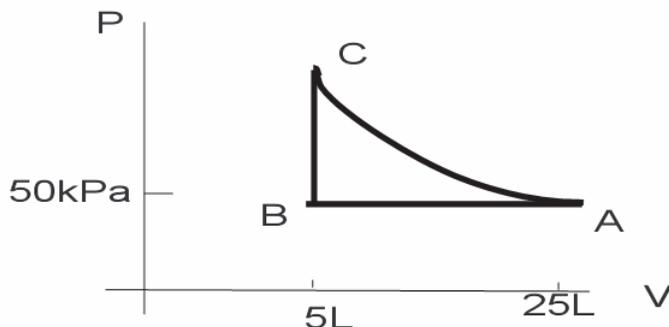


FIG. 1. The gas goes in a loop ABCA, where the section AC is isothermal.

We know the  $(P, V)$  coordinates of A, namely  $P_A = 50 \text{ kPa}$  and  $V_A = 25 \text{ L}$ . We also know that CA is an isotherm which means that A and C are at the same temperature. Since the number of moles of gas isn't changing in the step AC, we know that the quantity  $PV$  is constant.

$$P_A V_A = P_C V_C$$

But since we know that  $V_C = 5 \text{ L}$ , then:

$$P_C = \frac{P_A V_A}{V_C} = \frac{(50 \text{ kPa})(25 \text{ L})}{5 \text{ L}} = 250 \text{ kPa}$$

Now that we know the  $(P, V)$  coordinates of all three points, we can calculate the work done by the gas in each step. The first branch is from point A to B. This is an isobaric process and assuming the gas is ideal:

$$W_{A \rightarrow B} = P(V_B - V_A) = (50 \text{ kPa})(5 \text{ L} - 25 \text{ L}) \left( 10^{-3} \frac{\text{m}^3}{\text{L}} \right) = -1000 \text{ J}$$

The minus sign here indicates that work is done on the gas in going from point A to point B. The step that goes from B to C is at constant volume and does no work.

$$W_{B \rightarrow C} = 0 \text{ J}$$

As previously mentioned, the step going from C to A is isothermal and so the work done by the gas is:

$$W_{C \rightarrow A} = P_A V_A \ln \left( \frac{V_A}{V_C} \right) = (50 \text{ kPa})(25 \text{ L}) \left( 10^{-3} \frac{\text{m}^3}{\text{L}} \right) \ln \left( \frac{25 \text{ L}}{5 \text{ L}} \right) \approx 2012 \text{ J}$$

So the total work is:

$$W_{ABCA} = W_{A \rightarrow B} + W_{B \rightarrow C} + W_{C \rightarrow A} \approx -1000 \text{ J} + 0 \text{ J} + 2012 \text{ J} = 1012 \text{ J}$$

Since the system returns to its original state, its internal energy doesn't have a net change. This means that any work done by the gas must have been absorbed as heat from its surroundings. So the heat absorbed in the full cycle is about  $1012 \text{ J}$ .

11. One mole of a gas with  $\gamma = 4/3$  goes over the cycle ABCA as in Figure 2 where one of AB or AC is isothermal and the other adiabatic. (You figure out which.) Write down the  $(P, V, T)$  coordinates of A, B and C (some of which are already given). What is the work done in each part of the cycle and the heat absorbed or rejected in the full cycle?

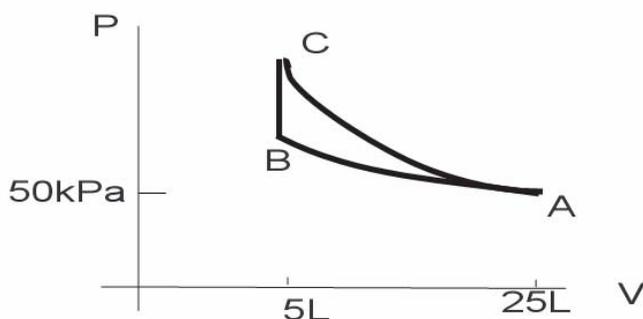


FIG. 2. The gas goes in a loop ABCA, where either AB or AC is isothermal and the other is adiabatic.

Let's first figure out which of AB and AC is the isothermal and which is the adiabatic step. Along the isothermal step we know that:

$$P_A V_A = P_{\text{isothermal}} V_{\text{isothermal}}$$

But since the pressure and volume is known for point A, and the volumes of both points B and C are known and the same:

$$P_{\text{isothermal}} = \frac{P_A V_A}{V_B} = \frac{(50\text{kPa})(25L)}{5L} = 250\text{kPa}$$

Along the adiabatic step we know that:

$$P_A V_A^\gamma = P_{\text{adiabatic}} V_{\text{adiabatic}}^\gamma$$

which can be solved for the pressure resulting from the adiabatic step:

$$P_{\text{adiabatic}} = P_A \left( \frac{V_A}{V_{\text{adiabatic}}} \right)^\gamma = (50\text{kPa}) \left( \frac{25L}{5L} \right)^{4/3} \approx 427.5\text{kPa}$$

Since we can see from the diagram that point C has a higher pressure than point B we know that AC must be the adiabatic process and AB must be the isothermal process.

Let's now track the work done in each part of the cycle. Along the isothermal path from A to B:

$$W_{A \rightarrow B} = -P_A V_A \ln \left( \frac{V_A}{V_B} \right) = -(50\text{kPa})(25L)(10^{-3} \text{ m}^3/\text{L}) \ln \left( \frac{25L}{5L} \right) \approx -2012\text{J}$$

The minus sign here indicates that work is done on the gas in going from point A to point B. The step that goes from B to C is at constant volume and so does no work.

$$W_{B \rightarrow C} = 0\text{J}$$

The step that goes from C to A is adiabatic and does work:

$$W_{C \rightarrow A} = \frac{P_C V_C - P_A V_A}{\gamma - 1} = \frac{(427.5 \text{ kPa})(5 \text{ L})(10^{-3} \text{ m}^3/\text{L}) - (50 \text{ kPa})(25 \text{ L})(10^{-3} \text{ m}^3/\text{L})}{\frac{4}{3} - 1} \approx 2662 \text{ J}$$

So the total work is:

$$W_{ABCA} = W_{A \rightarrow B} + W_{B \rightarrow C} + W_{C \rightarrow A} \approx -2012 \text{ J} + 0 \text{ J} + 2662 \text{ J} = 650 \text{ J}$$

Since the system returns to its original state, its internal energy doesn't have a net change. This means that any work done by the gas must have been absorbed as heat from its surroundings. So the heat absorbed in the full cycle is about 650 J.

To complete the  $(P, V, T)$  coordinates for the three points, we can make use of the fact that  $T_A = T_B$  since they are connected by an isothermal process. To get values for  $T_A$  and  $T_C$  we just make use of the ideal gas law:

$$T = \frac{PV}{nR}$$

So, using the previously found values:

$$T_A = \frac{(50 \text{ kPa})(25 \text{ L})(10^{-3} \text{ m}^3/\text{L})}{(1 \text{ mol}) \left( 8.314 \frac{\text{J}}{\text{mol} \cdot \text{K}} \right)} \approx 150 \text{ K}$$

$$T_C = \frac{(427.5 \text{ kPa})(5 \text{ L})(10^{-3} \text{ m}^3/\text{L})}{(1 \text{ mol}) \left( 8.314 \frac{\text{J}}{\text{mol} \cdot \text{K}} \right)} \approx 257 \text{ K}$$

Which finally lets us write the final  $(P, V, T)$  coordinates for the three points.

Point A: (50 kPa, 25 L, 150 K)

Point B: (250 kPa, 5 L, 150 K)

Point C: (427.5 kPa, 5 L, 257 K)