Physics 200
Problem Set 10
Solution

1. A vertical tube of radius 1 cm, open at the top to the atmosphere, contains 2 cm of oil ($\rho_{\text{oil}} = 0.82\rho_{\text{water}}$) floating on 3 cm of water. What is the gauge pressure (pressure in excess of atmospheric) at the bottom?

   Answer: Only the depth matters, so that

$$\begin{aligned} p_{\text{gauge}} &= \rho_{\text{oil}} g h_{\text{oil}} + \rho_{\text{water}} g h_{\text{water}} = \rho_{\text{water}} g (0.82 h_{\text{oil}} + h_{\text{water}}) \\ &= (1000\ \text{kg/m}^3)(9.8\ \text{m/s}^2)[0.82(0.02\ \text{m}) + 0.03\ \text{m}] = 460\ \text{Pa}. \end{aligned}$$

2. A person wants to suck water through a straw 120 cm tall. What is the minimum pressure difference between the atmosphere and the inside of the person's mouth?

   Answer: The pressure difference must be at least

$$\rho g h = (1000\ \text{kg/m}^3)(9.8\ \text{m/s}^2)(1.2\ \text{m}) = 1.2 \times 10^4\ \text{Pa}.$$

3. A wide can of water is filled to height h. At what height from the base should I drill a hole to (i) get the maximum range for the jet that comes out, and (ii) ensure that the jet travels a horizontal distance equal to the vertical distance? (iii) What is the maximum horizontal distance the jet can travel?

   Answer: Since the can is wide, we can assume that the velocity of the water at the top of the can is zero. If the hole is drilled at height z from the base, then the (horizontal) velocity at the hole is determined by Bernoulli's equation

$$\rho g(h-z) = \frac{1}{2}\rho v^2 \implies v = \sqrt{2g(h-z)}.$$

   Going back to projectile motion for a moment, a particle dropped from height z takes time $t = \sqrt{2z/g}$ to reach the ground, so it travels a distance $R = vt$ in the horizontal direction. Plugging in the above then gives

$$R = \sqrt{2g(h-z)}\sqrt{\frac{2z}{g}} = 2\sqrt{z(h-z)}.$$

   i) To make life easy we can maximize the square of R:

$$\frac{1}{4}\frac{d(R^2)}{dz} = \frac{d}{dz}[z(h-z)] = h - 2z = 0 \implies z = \frac{h}{2}.$$

   ii) This time we solve for

$$R = 2\sqrt{z(h-z)} = z \implies 5z^2 = 4zh,$$

   which means $z = \frac{4}{5}h$. Note that the $z = 0$ solution just means the water comes out at the bottom so it doesn't travel at all before hitting the ground.

   iii) Plugging in the value we found in part (i) into the equation for R, we find that the maximum range is

$$R_{\max} = h.$$

4. A rectangular hole is made on the side of a cylindrical container of water. The hole has width w and its upper and lower edges (parallel to the base of the cylinder) are at depths d_1 and d_2 respectively measured from the water level.

   i) Show using integration that the initial flow rate out of this hole is

   $$\frac{2w}{3}\sqrt{2g}\left[d_2^{3/2} - d_1^{3/2}\right].$$

   Neglect the motion of water at the top of the tank.

   ii) Suppose $d_2 = d_1 + \delta$, where δ is very small, so that we can ignore the variation of the depth from the top to the bottom of the hole. State what you would expect the rate to be in this case and show that the exact answer above reduces to it in this limit. (Remember $(1+x)^n = 1 + nx + \ldots$)

   Answer:

   i) The final answer looks complicated at first, but in fact it is just the result of integrating v. If we neglect the motion of water at the top, then Bernoulli's equation tells us

   $$\rho g z = \frac{1}{2}\rho v^2 \implies v = \sqrt{2gz},$$

   where z is the depth below the water. For any small increment of height dz the rate of water flowing out is $vw\,dz$ (velocity times area; check the units), so integrating gives us the flow rate

   $$\frac{dV}{dt} = w\int_{d_1}^{d_2} v\,dz = w\sqrt{2g}\int_{d_1}^{d_2}\sqrt{z}\,dz = \frac{2w}{3}\sqrt{2g}\left(d_2^{3/2} - d_1^{3/2}\right).$$

   ii) In this case we should not have to integrate at all, and we expect that we can simply apply Bernoulli's equation at one point to get

   $$\frac{dV}{dt} = vA = \sqrt{2gd_1}\,w\delta.$$

   We can confirm this from the answer above, keeping in mind that when we say "δ is small," what we really mean is that δ is small compared to d_1, or $\delta/d_1 \ll 1$ (anything we call small must be unitless). The expansion $(1+x)^\alpha \simeq 1 + \alpha x$ for small x tells us

   $$\begin{aligned}\frac{dV}{dt} &= \frac{2w}{3}\sqrt{2g}\left[(d_1+\delta)^{3/2} - d_1^{3/2}\right] = \frac{2w}{3}\sqrt{2gd_1^3}\left[\left(1+\frac{\delta}{d_1}\right)^{3/2} - 1\right] \\ &\simeq \frac{2w}{3}\sqrt{2gd_1}\,d_1\left(1 + \frac{3}{2}\frac{\delta}{d_1} - 1\right) = \sqrt{2gd_1}\,w\delta.\end{aligned}$$

5. A piece of cork of density $\rho = .22\rho_{\text{water}}$ is held down fully submerged in water by a thread anchored at the base of the container. What is T/mg, where T is the tension of the thread and mg is the mass of the piece of cork?

   Answer: Let the volume of the cork be V, so that the mass of the cork is $m = \rho V$. The three forces on the cork are the force of gravity mg downward, the tension T downward, and the buoyant force $\rho_{\text{water}}Vg = mg/0.22$ upward. If the cork is held in place then

   $$\frac{mg}{0.22} - mg - T = 0 \implies \frac{T}{mg} = \frac{1}{0.22} - 1 = 3.5.$$

6. A cylinder of cross section A and mass m floats vertically in a fluid of density ρ (in other words, the cross section is parallel to the surface). If disturbed from this equilibrium, what will be the period of small (vertical) oscillations?

   Answer: We can choose the pressure at the equilibrium point to be zero, so that when the cylinder is a distance z above this point the pressure is $-\rho g z$ and hence the force $-\rho g z A$. This causes the cylinder to accelerate, so that (assume the cylinder is thin, so we can use the pressure at one depth)

   $$m\frac{d^2z}{dt^2} = -\rho g z A \implies \frac{d^2z}{dt^2} = -\left(\frac{\rho g A}{m}\right) z.$$

   We can immediately read off the angular frequency ω from this equation, and the period is

   $$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{\rho g A}}.$$

7. A horizontal pipe of diameter $d_1 = 10$ cm carrying water has a constriction of diameter $d_2 = 4$ cm. If $P_1 = 10^5$ Pa and $P_2 = 8\times10^4$ Pa, what is the flow rate?

   Answer: Bernoulli's equation with equal heights gives

   $$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2,$$

   and the conservation of volume gives $v_1 A_1 = v_2 A_2$. Substituting, we get

   $$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho\left(\frac{v_1 A_1}{A_2}\right)^2 \implies P_1 - P_2 = \frac{1}{2}\rho v_1^2\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right],$$

   which means

   $$v_1 = \sqrt{\frac{2(P_1 - P_2)/\rho}{(A_1/A_2)^2 - 1}}$$

   and the flow rate is

   $$v_1 A_1 = \sqrt{\frac{2(P_1-P_2)/\rho}{1/A_2^2 - 1/A_1^2}} = \frac{\pi}{4}\sqrt{\frac{2(P_1-P_2)/\rho}{1/d_2^4 - 1/d_1^4}} = \frac{\pi}{4}\sqrt{\frac{2(10^5\text{ Pa} - 8\times10^4\text{ Pa})/(1000\text{ kg/m}^3)}{1/(0.04\text{ m})^4 - 1/(0.1\text{ m})^4}} = 8.1\times10^{-3}\text{ m}^3/\text{s}.$$

8. Find the lift on an airplane wing of area 50 m^2 if the velocity of air at the upper and lower parts is 90 m/s and 70 m/s respectively. Assume $\rho_{\text{air}} = 1.16\text{ kg/m}^3$.

   Answer: The lift is the difference in pressure between the top and bottom of the wing, multiplied by the area of the wing. From Bernoulli's equation (we assume the wing is very thin)

   $$L = (p_1 - p_2)A = \frac{1}{2}\rho_{\text{air}}(v_2^2 - v_1^2)A = \frac{1}{2}(1.16\text{ kg/m}^3)[(90\text{ m/s})^2 - (70\text{ m/s})^2](50\text{ m}^2) = 9.3\times10^4\text{ N}.$$

9. A wall of an outdoor pool is 10 m long and 2 m deep. What force is needed to keep it in place?

   Answer: We assume the atmospheric pressure is common to both sides of the wall of the pool. Then the force on a thin strip of the wall is $\rho g z w\, dz$, so that

   $$F = \rho g w\int_0^d z\, dz = \frac{1}{2}\rho g w d^2 = \frac{1}{2}(1000\text{ kg/m}^3)(9.8\text{ m/s}^2)(10\text{ m})(2\text{ m})^2 = 196\text{ kN}.$$