- The power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely within the interval of convergence $|x| < R$, where

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

- Relations between functions will be satisfied order by order when they are replaced by their power series. You must know how to expand functions of functions, out to some desired order within the common interval of convergence.

- The power series representing a function may be integrated term by term and differentiated term by term within the interval of convergence to obtain the series for the integral or derivative of the function in question.

5

COMPLEX NUMBERS

5.1. Introduction

Let us consider the quadratic function $f(x) = x^2 - 5x + 6$ and ask where it vanishes. If we plot it against x, we will find that it vanishes at $x = 2$ and $x = 3$. This is also clear if we write f in factorized form as $f(x) = (x-2)(x-3)$. We could equivalently use the well-known formula for the roots $x_\pm$ of a quadratic equation:

$$ax^2 + bx + c = 0 \tag{5.1.1}$$

namely

$$x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{5.1.2}$$

to find the roots $x_\pm = 2, 3$. Suppose instead we consider

$$x^2 + x + 1 = 0. \tag{5.1.3}$$

A plot will show that this function is always positive and does not vanish for any point on the x-axis. We are then led to conclude that this quadratic equation has no roots. Let us pass from the graphical procedure which gives no solution to the algebraic one which does give some form of answer even now. It says

$$x_\pm = \frac{-1 \pm \sqrt{-3}}{2}. \tag{5.1.4}$$

The problem of course is that we do not know what to make of $\sqrt{-3}$ since there is no real number whose square is -3. Thus if we take the stand that a number is not a number unless it is a real number, we will have to conclude that some quadratic equations have roots and other do not.

5.2. Complex Numbers in Cartesian Form

This is how it was for many centuries until the rather bold suggestions was made that we admit into the fold of numbers also those of the form $\sqrt{-3}$. *All we need to know to do this is a set of consistent rules for manipulating such numbers; being able to visualize them is no prerequisite.* It is really up to us make up the rules since these entities have come out of the blue. The rules must, however, be free of contradictions. Of course, all this is pointless if the whole enterprise does nothing but merely exist. In the present case the idea has proven to be a very seminal one and we will see some of the evidence even in this elementary treatment.

We are dealing here with a case of mathematical abstraction or generalization, an example of which you have already seen, when we extended the notion of powers from integers to all real values, and examples of which you will see more than once in this course, say when we extend the notion of vectors in three dimensions with real components (which we can readily imagine) to vectors in any number of dimensions, or vectors with complex components (like $\sqrt{-3}$) which we cannot visualize. A general guideline when embarking on such generalizations is that we impose on the new entities as many properties of the more familiar entities as is possible. For example, when we passed from integer powers a^n to arbitrary powers a^x (whatever that meant) we demanded that noninteger powers obey the same rule of composition, i.e., $a^x a^y = a^{x+y}$ for all x and y.

Returning to our problem we will first demand that square roots of negative numbers (whatever they mean) still obey the rule that $\sqrt{ab} = \sqrt{a}\sqrt{b}$. Thus $\sqrt{-3} = \sqrt{3}\sqrt{-1}$. *The point of this is that the problem of taking the square root of any negative number is reduced to taking the root of* -1. Thus the basic building block we need to introduce, called the unit imaginary number, is

$$i = \sqrt{-1}. \tag{5.2.1}$$

In terms of i, the answer to Eqn. (5.1.3) is

$$x_{\pm} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i. \tag{5.2.2}$$

We will postulate that i will behave like a real number in all manipulations involving addition and multiplication and that the only new feature it will have is the one that defined it, namely that its square equals -1.

We now introduce a general complex number

$$z = x + iy \tag{5.2.3}$$

and refer to x and y as its *real and imaginary parts* and denote them by the symbols Re z and Im z. *A number with just $y \neq 0$ is called a pure imaginary number.* The solution to our quadratic equation has a real part $x = -\frac{1}{2}$ and an imaginary part $y = \pm\sqrt{3}/2$. We think of $x + iy$ as a *single number*. Indeed, if the number inside the radical had been 3 instead of -3, surely we would have treated say $-\frac{1}{2} + \sqrt{3}/2$ as a single number, as one of the roots. The same goes for $-\frac{1}{2} + i\sqrt{3}/2$.

The rules obeyed by complex numbers are as follows. Given two of them z_1 and z_2,

$$\begin{aligned} z_1 &= x_1 + iy_1 && (5.2.4)\\ z_2 &= x_2 + iy_2, \quad \text{we define,} && (5.2.5)\\ z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2) \quad \text{addition rule} && (5.2.6)\\ z_1 z_2 &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \quad \text{multiplication rule} && (5.2.7) \end{aligned}$$

where in the last equation we have opened out the brackets as we do with real numbers and used $i^2 = -1$. If we use these rules, we can verify that $x_{\pm} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ indeed satisfies Eqn. (5.1.1). Note that $z_1 z_2 = z_2 z_1$.

Problem 5.2.1. *Verify that this is so.*

It was emphasized that we must think of $z = x + iy$ as a single number. *However it is a single number which has two parts, which can be uniquely identified.* Thus although $7 = 5 + 2$ is a single number, the decomposition of 7 into 2 and 5 is not unique. On the other hand $z = 3 + 4i$ has a real part 3 and an imaginary part 4 and we cannot move things back and forth between the real and imaginary parts keeping the number fixed. *Thus if two complex numbers are equal, their real parts and imaginary parts are separately equal:*

$$\begin{aligned} z_1 = x_1 + iy_1 &= z_2 = x_2 + iy_2 \text{ implies} && (5.2.8)\\ x_1 &= x_2 && (5.2.9)\\ y_1 &= y_2. && (5.2.10) \end{aligned}$$

Suppose this were not true. This would imply $x_1 - x_2 = i(y_2 - y_1)$, without both of them vanishing separately. Squaring both sides, we would find a positive definite left-hand side and a negative definite right-hand side. The only way to avoid a contradiction is for both sides to vanish, giving us $0 = -0$, which is something we can live with.

Now, given any real number x, we can associate with it a unique number $-x$, called its negative. We can do that with a complex number $z = x + iy$ too, by negating x and y. This number is called $-z$. But now we have an intermediate choice in which we negate just y: the result

$$z^* = x - iy \tag{5.2.11}$$

pronounced "z-star" is called the *complex conjugate* of z. Some people like to write it $\bar{z}$ and call it "z-bar".

Note that

$$zz^* = x^2 + y^2 \equiv |z|^2 \geq 0. \tag{5.2.12}$$

One refers to $|z| = \sqrt{x^2+y^2}$ as the *modulus* or *absolute value* of the complex number z. It is useful to know that given z and its z^*, we can recover the real and imaginary parts of z as follows:

$$\text{Re } z \equiv x = \frac{z+z^*}{2} \tag{5.2.13}$$

$$\text{Im } z \equiv y = \frac{z-z^*}{2i} \tag{5.2.14}$$

Note that we did not ever explicitly define the rules for division of complex numbers. This is because we can carry out division as the inverse of multiplication. Thus if $z_1/z_2 = z_3$, we can multiply both sides by z_2 (which we know how to do) and solve for x_3 and y_3 by equating real and imaginary parts in

$$x_1 + iy_1 = (x_3 + iy_3)(x_2 + iy_2) \tag{5.2.15}$$

to obtain (upon solving a pair of simultaneous equations)

$$x_3 = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} \tag{5.2.16}$$

$$y_3 = \frac{y_1x_2 - y_2x_1}{x_2^2 + y_2^2}. \tag{5.2.17}$$

This result is more easily obtained by using the notion of complex conjugates. First note that

$$\frac{1}{z} = \frac{z^*}{zz^*} \tag{5.2.18}$$

$$= \frac{x - iy}{x^2+y^2}. \tag{5.2.19}$$

Applying this result to the $1/z_2$ in the ratio z_1/z_2, we can obtain Eqns. (5.2.16, 5.2.17) more easily than before. In other words, by invoking the complex conjugate we have reduced the problem to division by a *real* number, namely z^*z, which is a familiar concept.

Complex conjugation can be viewed as the process of replacing i *by* $-i$ *within the complex number.* Stated this way it is clear that the complex conjugate of a product is the product of the complex conjugates

$$(z_1z_2)^* = z_1^*z_2^* \tag{5.2.20}$$

as you may check by explicit evaluation of both sides. The same obviously is true for the sum of two complex numbers. Now, if two complex numbers are

equal, their real imaginary parts are separately equal. Consider an equation with complex numbers on both sides. *If we replace all the numbers by their conjugates, the resulting quantities must still be equal.* This is because the imaginary parts, originally equal on both sides, will continue to be equal after signs are changed on both sides. Thus every complex equation implies another one obtained by complex conjugation of both sides. The latter does not contain any more or less information: both tell us the real and imaginary parts are separately equal.

Let us consider an illustrative example. Let

$$z_1 = \frac{\sqrt{2}+i}{1-i} \qquad z_2 = \frac{1+i}{\sqrt{2}} \tag{5.2.21}$$

Let us first simplify z_1 to the form $x + iy$ by multiplying the numerator and denominator by the complex conjugate of the latter:

$$z_1 = \frac{(\sqrt{2}+i)(1+i)}{(1-i)(1+i)} = \frac{(\sqrt{2}-1)+i(\sqrt{2}+1)}{2} \tag{5.2.22}$$

Since z_2 is already in this form, let us move on to compute

$$z_1z_2 = \frac{(\sqrt{2}-1)+i(\sqrt{2}+1)}{2}\cdot\frac{1+i}{\sqrt{2}} \tag{5.2.23}$$

$$= \frac{(\sqrt{2}-1)+i(\sqrt{2}+1)+i(\sqrt{2}-1)+i^2(\sqrt{2}+1)}{2\sqrt{2}} \tag{5.2.24}$$

$$= i - \frac{1}{\sqrt{2}} \tag{5.2.25}$$

and

$$\frac{z_1}{z_2} = \frac{(\sqrt{2}-1)+i(\sqrt{2}+1)}{\sqrt{2}(1+i)} \tag{5.2.26}$$

$$= \frac{[(\sqrt{2}-1)+i(\sqrt{2}+1)][1-i]}{2\sqrt{2}} \tag{5.2.27}$$

$$= 1 + \frac{i}{\sqrt{2}}. \tag{5.2.28}$$

Problem 5.2.2. *Show that* $z_2^2 = i$.

The following exercises should give you some more practice with the manipulation of complex numbers.

Problem 5.2.3. *Solve for* x *and* y *given*

$$\frac{2+3i}{6+7i} + \frac{2}{x+iy} = 2 + 9i.$$

Problem 5.2.4. *Find the real part, imaginary part, modulus, complex conjugate, and inverse of the following numbers: (i)* $\frac{2}{3+4i}$, *(ii)* $(3+4i)^2$, *(iii)* $\frac{3+4i}{3-4i}$, *(iv)* $\frac{1+\sqrt{2}i}{1-\sqrt{3}i}$, *and (v)* $\cos\theta + i\sin\theta$.

Problem 5.2.5. *Show that a polynomial with real coefficients has only real roots or complex roots that come in complex conjugate pairs.*

Problem 5.2.6. (Very important). *Prove algebraically that*

$$|\,\mathrm{Re}\ z| \ \leq\ |z| \tag{5.2.29}$$
$$|\,\mathrm{Im}\ z| \ \leq\ |z|. \tag{5.2.30}$$

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1 z_2^*) \tag{5.2.31}$$

$$|z_1 + z_2| \leq |z_1| + |z_2| \tag{5.2.32}$$

$$|z_1 z_2| = |z_1||z_2| \tag{5.2.33}$$

Problem 5.2.7. (Important). *Verify that the numbers* z_1, z_2 *from Eqn. (5.2.21) respect Eqns. (5.2.31-5.2.33).*

Recall that all real numbers can be visualized as points on a line, called the x axis. To visualize all complex numbers we introduce the *complex plane* which is just the $x-y$ plane. The complex number $z = x + iy$ is labeled as shown in Fig. 5.1. The conjugate is z^*. The significance of r and θ will now be explained.

5.3. Polar Form of Complex Numbers

We begin this section with a remarkable identity due to Euler:

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{5.3.1}$$

where we will choose θ to be real. To prove this identity, we must define what we mean by e raised to a complex power $i\theta$. We define $e^{anything}$ to be the infinite power series associated with the exponential function e^x with x replaced by $anything$. Thus

$$e^{elephant} = \sum_{n=0}^{\infty} \frac{(elephant)^n}{n!} \tag{5.3.2}$$

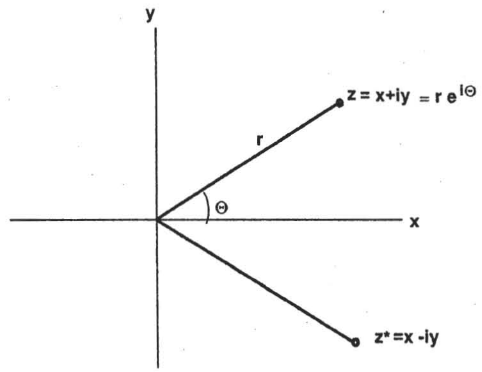

Figure 5.1. The complex plane.

which converges for any finite sized $elephant$.

Turning to our problem, we expand the infinite series for the exponential and collect the real and imaginary parts as follows:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \tag{5.3.3}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n(\theta)^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{(-1)^n(\theta)^{2n+1}}{(2n+1)!} \tag{5.3.4}$$
$$= \cos\theta + i\sin\theta \tag{5.3.5}$$

where we have used the fact that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and so on, as well as the infinite series that define the sine and cosine functions. (These expansions converge for all finite θ, as shown before. The presence of i does not in any way complicate the question of convergence since it either turns into a ± 1 or into $\pm i$.) Setting $\theta = \pi$ we obtain one of the most remarkable formulae in mathematics:

$$e^{i\pi} + 1 = 0. \tag{5.3.6}$$

Who would have thought that π which enters as the ratio of circumference to diameter, e, as the natural base for logarithms, i, as the fundamental imaginary unit and 0 and 1 (which we know all about from infancy) would all be tied together in any way, not to mention such a simple and compact way? I hope I never stumble into anything like this formula, for nothing I do after that in life would have any significance.

Look at Fig. 5.1 of the complex plane and note that

$$z \ = \ x + iy \tag{5.3.7}$$

$$
\begin{aligned}
&= r\left[\frac{x}{r}+i\frac{y}{r}\right] && (5.3.8)\\
&= r\left[\cos\theta+i\sin\theta\right] && (5.3.9)\\
&= re^{i\theta}. && (5.3.10)
\end{aligned}
$$

The last equation is called the *polar form* of the complex number as compared to the *cartesian form* we have been using so far. One refers to θ as the *argument* or *phase* of the number and r as its modulus or absolute value. It is just as easy to visualize the number in the complex plane given the polar form as it was with the cartesian form. (As is true with polar coordinates in any context, θ is defined only modulo 2π, that is to say adding 2π to it changes nothing. We can usually restrict it to the interval $[0-2\pi]$.) All manipulations we did before in the cartesian form can of course be carried out in polar form, though some become easier and some harder. Thus if

$$
\begin{aligned}
z &= re^{i\theta}\text{ then} && (5.3.11)\\
z^* &= re^{-i\theta} && (5.3.12)\\
zz^* &= r^2 \quad (|z|=r) && (5.3.13)\\
\frac{1}{z} &= \frac{1}{r}e^{-i\theta} && (5.3.14)\\
z_1z_2 &= r_1r_2e^{i(\theta_1+\theta_2)}. && (5.3.15)
\end{aligned}
$$

(We are using the fact that the law of composition of exponents under a product works for complex exponents as well. Indeed this is built into the exponential function defined by the infinite series. You may check that this works to any given order even for imaginary arguments. In Chapter 6, this will be proven more directly.) The last formula tells us how easy it is to multiply or divide two complex numbers in polar form:

To multiply two complex numbers, multiply their moduli and add their phases. To divide, divide by the modulus and subtract the phase of the denominator.

On the other hand to add two complex numbers we have to go back to the cartesian form, add the components and revert to the polar form.

Let us return to Eqn. (5.2.21) and manipulate the numbers in polar form. First

$$
\begin{aligned}
z_1 &= \frac{(\sqrt{2}-1)+i(\sqrt{2}+1)}{2} && (5.3.16)\\
&= \sqrt{\frac{(\sqrt{2}-1)^2+(\sqrt{2}+1)^2}{4}}\exp\left[i\arctan\left[\frac{1+\sqrt{2}}{\sqrt{2}-1}\right]\right] && (5.3.17)\\
&= \sqrt{\frac{3}{2}}e^{1.400i}. && (5.3.18)
\end{aligned}
$$

As for z_2,

$$
z_2=\sqrt{\frac{1}{2}+\frac{1}{2}}e^{i\arctan 1}=e^{i\pi/4}=e^{.785i}. \qquad (5.3.19)
$$

Now it is easy to form the product and quotient

$$
\begin{aligned}
z_1z_2 &= \sqrt{\frac{3}{2}}\cdot 1e^{(1.400+.785)i} && (5.3.20)\\
&= \sqrt{\frac{3}{2}}e^{2.185i}=1.224\ (\cos 2.185+i\sin 2.185) && (5.3.21)\\
&= -\frac{1}{\sqrt{2}}+i && (5.3.22)\\
\frac{z_1}{z_2} &= \sqrt{\frac{3}{2}}e^{(1.400-.785)i}=1.224\ (\cos .615+i\sin .615) &&\\
&= 1+\frac{i}{\sqrt{2}} && (5.3.23)
\end{aligned}
$$

in agreement with the calculation done earlier in cartesian form.

Complex numbers $z=re^{i\theta}$ with $r=1$ have $|z|=1$ and are called *unimodular*. We may imagine them as lying on a circle of unit radius in the complex plane. Special points on this circle are

$$
\begin{aligned}
\theta &= 0\ (1) && (5.3.24)\\
&= \pi/2\ (i) && (5.3.25)\\
&= \pi\ (-1) && (5.3.26)\\
&= -\pi/2\ (-i). && (5.3.27)
\end{aligned}
$$

You are expected to know these points at all times.

Problem 5.3.1. *Verify the correctness of the above using Euler's formula.*

When we work with real numbers, we know that multiplication by a number, say 4, rescales the given number by 4. *Multiplying a number in the complex plane by $re^{i\theta}$, rescales its length (or modulus) by r and also rotates it counterclockwise by θ.* Multiplying by a unimodular number simply rotates without any rescaling.

Problem 5.3.2. *For the following pairs of numbers, give their polar form, their complex conjugates, their moduli, product, the quotient z_1/z_2, and the complex conjugate of the quotient:*

$$
z_1=\frac{1+i}{\sqrt{2}} \quad z_2=\sqrt{3}-i
$$

$$
z_1=\frac{3+4i}{3-4i} \quad z_2=\left[\frac{1+2i}{1-3i}\right]^2.
$$

Problem 5.3.3. *Express the sum of the following in* polar form:

$$z_1 = 2e^{i\pi/4} \quad z_2 = 6e^{i\pi/3}.$$

Recall from Euler's formula that for real θ

$$\cos\theta = \text{Re } e^{i\theta} = \frac{e^{i\theta}+e^{-i\theta}}{2} \tag{5.3.28}$$

$$\sin\theta = \text{Im } e^{i\theta} = \frac{e^{i\theta}-e^{-i\theta}}{2i}. \tag{5.3.29}$$

You should remember the above results at all times.

Problem 5.3.4. *Check the following familiar trigonometric identities by expressing all functions in terms of exponentials:* $\sin^2 x + \cos^2 x = 1$, $\sin 2x = 2\sin x \cos x$, $\cos 2x = \cos^2 x - \sin^2 x$.

Problem 5.3.5. *Consider the series*

$$e^{i\theta} + e^{3i\theta} + \cdots + e^{(2n-1)i\theta}. \tag{5.3.30}$$

Sum this geometric series, take the real and imaginary parts of both sides and show that

$$\cos\theta + \cos 3\theta + \cdots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta}$$

and that a similar sum with sines adds up to $\sin^2 n\theta/\sin\theta$.

Problem 5.3.6. *Consider* De Moivre's Theorem, *which states that* $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. *This follows from taking the* n*-th power of both sides of Euler's theorem. Find the formula for* $\cos 4\theta$ *and* $\sin 4\theta$ *is terms of* $\cos\theta$ *and* $\sin\theta$. *Given* $e^{iA}e^{iB} = e^{i(A+B)}$ *deduce* $\cos(A+B)$ *and* $\sin(A+B)$.

5.5. Summary

Here are the highlights.

- If

$$z = x + iy,$$

then x is the real part or Re z, y is the imaginary part or Im z. This is the cartesian form of z.

- The polar form is

$$z = x + iy = re^{i\theta} \quad r = \sqrt{x^2+y^2}, \ \tan\theta = y/x,$$

where we have used Euler's identity:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

One calls r the modulus and θ the phase or argument of z.

- From Euler's theorem, by taking real and imaginary parts, we find

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

Forget these and you are doomed.

- The complex conjugate is

$$z^* = x - iy = re^{-i\theta}.$$

- The modulus squared is given by

$$|z|^2 = x^2 + y^2 = zz^*$$

- Some very useful relations and inequalities to remember:

$$|\text{Re } z| \le |z| \qquad |\text{Im } z| \le |z|.$$

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1 z_2^*)$$

$$|z_1 + z_2| \le |z_1| + |z_2| \qquad |z_1 z_2| = |z_1||z_2|$$

$$(z_1 z_2)^* = z_1^* z_2^*$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}.$$