

## Problem Set III Solutions

1. The block is at rest which means that  $F_x = F_y = 0$ . From Figure 1, it is clear that

$$F_x = ma_x = 0 \implies T_1 \cos a = T_2 \cos b \quad (1)$$

$$F_y = ma_y = 0 \implies T_1 \sin a + T_2 \sin b = mg. \quad (2)$$

Solving Equation 1 for  $T_2$  and then plugging back into Equation 2 to solve for  $T_1$ ,

$$T_2 = T_1 \frac{\cos a}{\cos b}$$

$$T_1 \left( \sin a + \frac{\cos a}{\cos b} \sin b \right) = mg \implies T_1 = \frac{mg}{\sin a + \cos a \tan b}.$$

Plugging in numbers:

$$T_1 = \frac{(10\text{kg})(9.8\text{m/s}^2)}{\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{3}} = 49\text{ N}$$

$$T_2 = T_1 \left( \frac{\sqrt{3}}{2} \right) 2 = 49\sqrt{3}\text{ N} = 85\text{ N}$$

2. (i) For the case where there is no friction between the block and the table, the force on the two block combination is

$$F = (M + m)a = -kA \implies a = \frac{-kA}{M + m}.$$

Assuming the block of mass  $m$  does not slip, both blocks accelerate at the same rate. The force that is accelerating the smaller block is the force due to friction. Intuitively, if there were no friction between the blocks, the smaller mass would stay at the stretched position and not be pulled back by the spring when the mass  $M$  was released. (It would then, of course, fall to the ground.) Therefore, in this case, friction must act in the direction of the motion in order to keep the smaller block on top of the larger block.  $ma = F_f \leq \mu mg$ . Since we want to know the maximum distance the block can be pulled, we want the maximum value for the friction. Using the value for acceleration that we found above,

$$ma = m \frac{kA}{m + M} \leq \mu mg$$

$$A_{max} = \frac{\mu g(m + M)}{k}$$

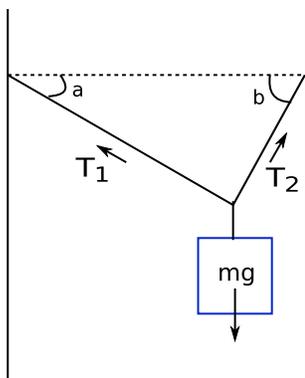


FIG. 1: Problem 1.

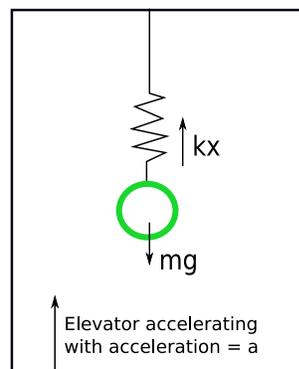


FIG. 2: Problem 4

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(ii) When you add friction between the block and the table, the initial equation for the acceleration changes to

$$F = (M + m)a = -kA + \mu'(m + M)g \implies a = \frac{-kA + \mu'(m + M)g}{M + m}.$$

Now the force you need to keep the little block from slipping is

$$ma = \left(\frac{m}{M + m}\right)(kA - \mu'(m + M)g) \leq \mu mg$$

$$A_{max} = \frac{(M + m)g(\mu + \mu')}{k}$$

3. The average force is the change in energy divided by the change in distance. Assuming the seat belt keeps the passenger safely in the seat,

$$\begin{aligned} \bar{F} &= \frac{\Delta(E)}{\Delta(x)} \\ &= \frac{1}{x_f - x_0} \left( \frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2 \right) \\ &= \frac{1}{0.94m} \left( \frac{-1}{2} \right) (75kg)(70km/h)^2 \\ &= \frac{1}{0.94m} \left( \frac{-1}{2} \right) (75kg) \left( 70 \frac{km}{h} \frac{1000m}{1km} \frac{1h}{3600s} \right)^2 \\ &= -15 kN \end{aligned}$$

4. There are two forces acting on the mass, the force due to gravity and the force from the spring (see Figure 2). Writing down the forces on the mass,

$$F = ma = kx - mg \implies x = \frac{m(a + g)}{k}.$$

The  $x$  I just solved for is the distance the spring stretches due to gravity and the acceleration of the elevator, but does not include its unstretched length. Therefore, the total length of the spring is given by

$$X = 80 \text{ cm} + \frac{m(a + g)}{k}. \quad (3)$$

(a) Plugging into Equation 3

$$X = 80 \text{ cm} + \frac{(7.2kg)(0.95 + 9.8)(m/s^2)}{150N/m} = 132 \text{ cm}$$

(b) In this case,  $a=0$ , so

$$X = 80 \text{ cm} + \frac{(7.2kg)(9.8m/s^2)}{150N/m} = 127 \text{ cm}$$

(c) We need to calculate the acceleration and plug it into Equation 3.

$$\begin{aligned} \bar{a} &= \frac{\Delta(v)}{\Delta(t)} = \frac{0 - 14m/s}{9.0s} \\ X &= 80 \text{ cm} + \frac{(7.2kg)\left(\frac{-14}{9} + 9.8\right)(m/s^2)}{150N/m} = 120 \text{ cm} \end{aligned}$$

(d) In this case, we are given  $X$  and need to solve for  $a$ .

$$X = 3.2 \text{ m} = 0.80 \text{ m} + \frac{(7.2kg)(a + 9.8m/s^2)}{150N/m} \implies a \leq 40.2 \text{ m/s}^2$$

To keep the mass from hitting the floor,  $a$  cannot be greater than  $40.2 \text{ m/s}^2$ . (This is assuming that the spring continues to obey the same force law when stretched to this length.)

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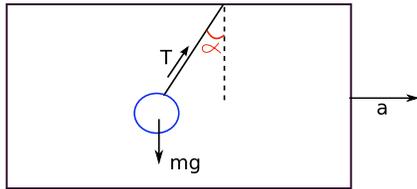


FIG. 3: Problem 5

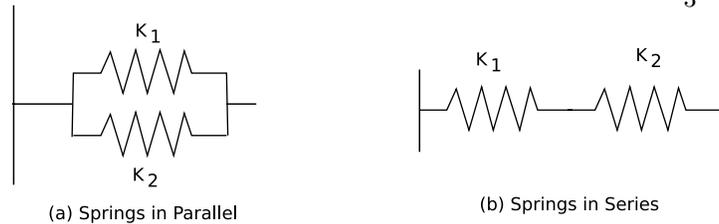


FIG. 4: Problem 6

5. The forces acting on the mass are the force due to gravity and the tension in the rope, see Figure 3. The mass is accelerating in the  $x$  direction, but not moving in  $y$ . Balancing the forces,

$$F_y = 0 = T \cos \alpha - mg \implies T = \frac{mg}{\cos \alpha}$$

$$F_x = ma = T \sin \alpha = \frac{mg \sin \alpha}{\cos \alpha} = mg \tan \alpha \implies \tan \alpha = \frac{a}{g}$$

So,

$$\alpha = \tan^{-1} \frac{a}{g}.$$

6. When the springs are connected in parallel (see Figure 4), they are both stretched or compressed the same amount  $x$ . Therefore, the force due to two springs in parallel is

$$F = F_1 + F_2 = -(k_1 x + k_2 x) = -(k_1 + k_2)x \implies k_e = (k_1 + k_2)$$

When the springs are connected in series, both springs feel the same force, but they are compressed different amounts. (They must feel the same force for the same reason the tension in a rope is constant throughout the rope. If the springs did not feel the same force, either they would have to be moving with respect to each other, or the system would break.)

$$F = F_1 = F_2$$

$$F = -k_1 x_1 = -k_2 x_2 = -k_e (x_1 + x_2)$$

Notice from the above equation that  $x_1 = -F/k_1$  and  $x_2 = -F/k_2$ . Plugging this into the equation for  $k_e$

$$F = k_e \left( \frac{F}{k_1} + \frac{F}{k_2} \right) \implies k_s = \frac{k_1 k_2}{k_1 + k_2}$$

7. You are spinning on a merry-go-round at a constant distance from the center. Your velocity is given by your distance divided by the time it takes you to travel that distance. In a time  $1/f$  seconds you make one full revolution, which is a distance of  $2\pi R$ . Your acceleration is given by  $a = v^2/R = (2\pi R f)^2/(R)$ . The force of friction is  $F_f \leq \mu_s mg$ . To find the minimum  $\mu_s$  to keep you from falling off, you want the force due to friction that will just balance your centripetal acceleration.

$$F = mR(2\pi f)^2 = \mu_s mg \implies \mu_s = \frac{R(2\pi f)^2}{g}$$

8. Let's start this problem by drawing a free body diagram for each mass, see Figure 5. The rope connecting the two masses carries a tension  $T$ . This tension must be the same throughout the rope otherwise the rope would break. Therefore, the force  $T$  acting on the small mass is the same as the force  $T$  acting on the large mass.

The mass  $M$  is not moving which means  $T = Mg$ . The mass  $m$  is moving in a circle of radius  $R$  on the table, so  $F = mv^2/R = T$ . Using this information,

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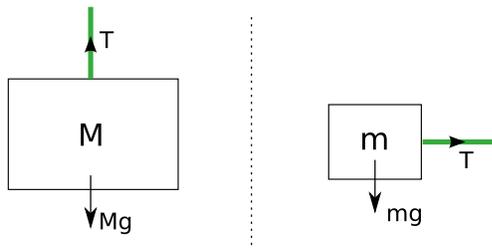


FIG. 5: Free body diagrams for the masses in problem 8.

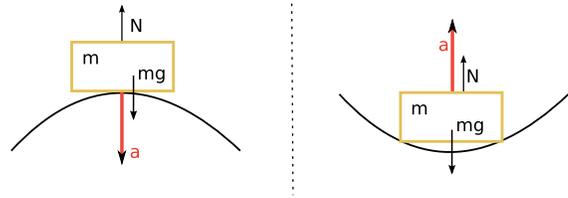


FIG. 6: Free body diagram that also shows the direction of acceleration for a car at the top and bottom of a hill.

- (a)  $T = Mg$   
 (b) The period is equal to  $1/f$ . Solving for  $f$  in the equation of motion for  $m$ ,

$$\frac{mv^2}{R} = mR(2\pi f)^2 = T = Mg \implies mR(2\pi f)^2 = Mg \implies f = \frac{1}{2\pi} \sqrt{\frac{Mg}{mR}}$$

That means that the period of revolution is  $2\pi \sqrt{\frac{mR}{Mg}}$ .

9. The only force holding the car to the road is gravity. Therefore, in order for the car to maintain contact with the road, the force due to gravity must be larger than or equal to the force required for circular motion. At the top of a hill, the acceleration is down, towards the center of the circle and the force due to gravity is also down, see Figure 6. The maximum speed the car can attain without leaving the road occurs when the normal force goes to zero.

$$mg = \frac{mv^2}{R} \implies v_{max} = \sqrt{gR}$$

Plugging in numbers,

$$v_{max} = \sqrt{(9.8m/s^2)(20m)} = 14m/s.$$

At the bottom of the hill, the force of gravity is still down, but the acceleration of the car is now up towards the center of the circle as shown in Figure 6. If the car is moving at the same speed as at the top of the hill, there has to be a large normal force to keep the car from crashing into the side of the hill. (This is what would happen if the car did not have enough centripetal acceleration to curve quickly enough with the road.)

$$\frac{mv^2}{R} = N - mg$$

Since the magnitude of the velocity is the same as at the top of the hill,  $mg = \frac{mv^2}{R}$  is still true. Therefore,  $N = 2mg = 19.6kN$ .

10. Since the lawnmower is moving at a constant velocity, there is no acceleration in either the  $x$  or  $y$  directions. From the free body diagram in Figure 7, the equations of motion are,

$$\begin{aligned} F_y = 0 &= N - mg - F_{ext} \sin \alpha \implies N = mg + F_{ext} \sin \alpha \\ F_x = 0 &= F_{ext} \cos \alpha - \mu N = F_{ext} \cos \alpha - \mu(mg + F_{ext} \sin \alpha) \implies F_{ext}(\cos \alpha - \mu \sin \alpha) - \mu mg = 0 \end{aligned}$$

Solving for  $F_{ext}$  and plugging in numbers,

$$F_{ext} = \frac{\mu mg}{\cos \alpha - \mu \sin \alpha} = \frac{(0.68)(22kg)(9.8m/s^2)}{\cos(35^\circ) - 0.68 \sin(35^\circ)} = 342N$$

The weight of the mower is only  $mg = (22kg)(9.8m/s^2) = 216N$ !

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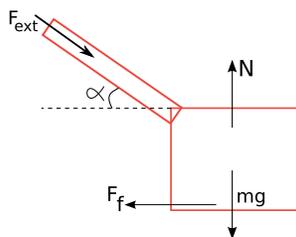


FIG. 7: Free body diagram for the lawnmower in problem 10.

11. When a roller coaster is at the top of it's track, it's minimum velocity occurs when the normal force equals 0. If the velocity gets any smaller than this, the roller coaster cannot maintain its circular motion and it falls off the track. The only force acting on the roller coaster is the force due to gravity.

$$\frac{Mv_{min}^2}{R} = Mg \implies v_{min} = \sqrt{Rg}$$

In this case,  $v = 2\sqrt{Rg}$ . When the roller coaster is at the top of the track, the normal force points away from the track, downwards towards the center of the circle. Solving for the normal force,

$$F = \frac{Mv^2}{R} = \frac{M(2\sqrt{Rg})^2}{R} = Mg + N \implies N = 3Mg$$

We can find the velocity of the roller coaster at the bottom of the track using conservation of energy. Taking the potential energy to be 0 at the bottom, at the top of the track, the roller coaster has kinetic energy equal to  $\frac{1}{2}Mv^2$  and a potential energy of  $Mg(2R)$ . This must be equal to its energy at the bottom of the track when it has only kinetic energy. Conserving energy and solving for the final velocity,

$$\begin{aligned} \frac{1}{2}Mv_f^2 &= \frac{1}{2}Mv^2 + Mg(2R) \\ &= \frac{1}{2}M(4Rg) + Mg(2R) \\ &= 4MgR \\ v_f &= \sqrt{8gR} \end{aligned}$$

Now, we need to find the normal force when the roller coaster is at the bottom of the track with velocity  $v_f$ . The normal force still points away from the track, but now that is upwards towards the center of the circle.

$$\frac{Mv_f^2}{R} = 8Mg = N - Mg \implies N = 9Mg$$

12. To solve this problem, we need to look at the equation of motion for the mass  $M$ . There is no problem taking the axes for each mass to be rotated in whatever direction is convenient, so I will choose my axes as shown in Figure 8. So long as the system is at rest,  $T = mg$ . The equations of motion for a stationary mass  $M$  are

$$F_y = 0 = N - Mg \cos \alpha \implies N = Mg \cos \alpha \quad (4)$$

$$F_x = 0 = Mg \sin \alpha - T \pm F_f = Mg \sin \alpha - mg \pm F_f \quad (5)$$

We have to choose the correct sign for  $F_f$  depending on whether the block is just about to slide down the hill or be pulled up the hill. The point at which  $M$  just starts to slide is when  $F_f = \mu_s N = \mu_s Mg \cos \alpha$ . Plugging in this value and solving equation 5 for  $m$  we find that when

$$m = M(\sin \alpha \pm \mu_s \cos \alpha)$$

$M$  just begins to slide.

- (i) If  $M$  is about to slide down the hill, then  $F_f$  is pointing in the negative direction. This means that for

$$m < M(\sin \alpha - \mu_s \cos \alpha)$$

$M$  will begin to slide down the hill as  $m$  is reduced from a value where the system is at rest.

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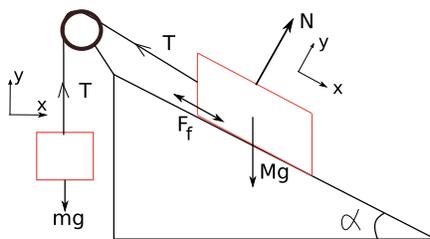


FIG. 8: Diagram for problem 12.

(ii) If  $M$  is about to be pulled up the hill, then  $F_f$  is pointing in the positive  $x$  direction. For

$$m > M(\sin \alpha + \mu_s \cos \alpha)$$

$M$  will be pulled up as  $m$  is increased from a value where the system is at rest.

For

$$M(\sin \alpha - \mu_s \cos \alpha) < m < M(\sin \alpha + \mu_s \cos \alpha), \quad (6)$$

if the system is initially at rest, it will stay at rest.

If, however, the system is given a push, it is possible that  $M$  will continue sliding up or down the hill even if  $m$  is in the range given by Equation 6. This is because  $\mu_k$ , the coefficient of kinetic friction, is generally smaller than  $\mu_s$ , the coefficient of static friction. Hence, once  $M$  starts moving, the force due to friction between the inclined plane and the block is smaller than when it was stationary.

The block will just be able to move if when it is given a push, the force of kinetic friction exactly balances the other forces acting on mass  $M$ .

$$\begin{aligned} Mg \sin \alpha &= T \mp F_{\text{kinetic friction}} = mg \mp \mu_k Mg \cos \alpha \\ m &= M(\sin \alpha \pm \mu_k \cos \alpha) \end{aligned}$$

If  $m < M(\sin \alpha - \mu_k \cos \alpha)$  and the system is given a push down the hill,  $M$  will continue moving down the hill. If  $m > M(\sin \alpha + \mu_k \cos \alpha)$  and the system is given a push up the hill,  $M$  will continue moving up the hill.

If you are trying to get  $M$  to go down the hill, the equation of motion is

$$F = Mg(\sin \alpha - \mu_s \cos \alpha) - mg \geq 0.$$

However, for some values of  $\alpha$  and  $\mu_s$  the quantity  $(\sin \alpha - \mu_s \cos \alpha)$  will never be greater than 0. This means that  $M$  will never slide down the hill by itself. The simplest intuitive example of this occurs when  $\alpha = 0$ . In this case, the  $x$  component of gravity acting on the mass  $M$  is 0, so  $M$  will never start pulling  $m$  up no matter how heavy it is. Conversely, if you are trying to pull  $M$  up the hill, the equation of motion is

$$F = Mg(\sin \alpha + \mu_s \cos \alpha) - mg \leq 0.$$

In this case, you can always make  $m$  big enough to have the total force be large enough to pull  $M$  up the hill.

(Please note that in solving this problem I assumed that either the system was stationary or just beginning to move. This means that  $mg = T$  by the equation of motion for  $m$ . However, once the system starts moving with some acceleration  $a$ ,  $ma = T - mg$  and the equation for the tension is no longer valid.)

13. To solve this problem we will use conservation of energy. When you pull a mass  $m$  connected to a spring with spring constant  $k$  a distance  $A$  from it's equilibrium point, the system has potential energy equal to  $\frac{1}{2}kA^2$ . We will use this information to answer the questions.

(i) At  $x = 0$ , the spring-mass system has only kinetic energy and no potential energy. Since there is no friction between the table and the mass, the energy now is the same as the energy was before you let the mass go. Setting the two energies equal and solving for  $v$ ,

$$\frac{1}{2}kA^2 = \frac{1}{2}mv^2 \implies v = A\sqrt{\frac{k}{m}}$$

- (ii) When the spring reaches its maximum compression, the system has no kinetic energy. Therefore,  $\frac{1}{2}kA^2 = \frac{1}{2}kx_f^2$ . Clearly,  $x_f = -A$ , or the block moves to a maximum distance  $A$  to the left of its equilibrium position.
- (iii) Now we add friction to the system. The work done by a constant frictional force is given by  $W_f = F_f d$  where  $d$  is the total distance the mass has traveled. In our case,  $F_f = \mu_k mg$ . Instead of all of the initial potential energy being converted into kinetic and potential energy, some of the initial energy is lost due to friction. The velocity at  $x = 0$  is now given by

$$\frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \mu_k mgA \implies v = A\sqrt{\frac{k}{m} - \frac{2\mu_k g}{A}}$$

- (iv) Solving for the distance the block travels to the left is a little trickier because  $W_f$  depends on the total distance the block travels. Take  $x_f$  to the furthest position the block moves to the left. We also know that  $x_f$  must be negative and we want the work done by friction to be a positive quantity, therefore I will take the total distance the block moves to be  $A - x_f$ . Then,

$$\begin{aligned}\frac{1}{2}kA^2 &= \frac{1}{2}kx_f^2 + \mu_k mg(A - x_f) \\ A^2 - x_f^2 &= \frac{2\mu_k mg}{k}(A - x_f) \\ (A - x_f)(A + x_f) &= \frac{2\mu_k mg}{k}(A - x_f) \\ x_f &= -A + \frac{2\mu_k mg}{k}\end{aligned}$$

So, if there is friction between the table and the mass, the mass will move to a distance  $x_f = A - \frac{2\mu_k mg}{k}$  to the left of the equilibrium point.

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