Physics 200a PSII

1. *Let* $\mathbf{A} = 3\mathbf{i} + 4\mathbf{j}$ *and* $\mathbf{B} = 5\mathbf{i} - 6\mathbf{j}$.
(i) Find $\mathbf{A}+\mathbf{B}$, $\mathbf{A}-\mathbf{B}$, $2\mathbf{A}+3\mathbf{B}$, *and* $\mathbf{C}$ *such that* $\mathbf{A}+\mathbf{B}+\mathbf{C} = \mathbf{0}$.
(ii) Find A *, the length of* $\mathbf{A}$ *and the angle it makes with the x-axis.*

$$\mathbf{A}+\mathbf{B} = 8\mathbf{i} - 2\mathbf{j}$$

$$\mathbf{A}-\mathbf{B} = -2\mathbf{i} + 10\mathbf{j}$$

$$2\mathbf{A}+3\mathbf{B} = 6\mathbf{i} + 8\mathbf{j} + 15\mathbf{i} - 18\mathbf{j} = 21\mathbf{i} - 10\mathbf{j}$$

$$\mathbf{C} = -\mathbf{A} - \mathbf{B} = -8\mathbf{i} + 2\mathbf{j}$$

$$(ii) A = \sqrt{3^2 + 4^2} = 5$$

2. *A train is moving with velocity* $\mathbf{v}_{TG} = 3\mathbf{i} + 4\mathbf{j}$ *relative to the ground. A bullet is fired in the train with velocity* $\mathbf{v}_{BT} = 15\mathbf{i} - 6\mathbf{j}$ *relative to the train. What is the bullets' velocity* $\mathbf{v}_{BG}$ *relative to the ground?*

$$\mathbf{v}_{BG} = \mathbf{v}_{BT} + \mathbf{v}_{TG} = 18\mathbf{i} - 2\mathbf{j}$$

3. *Consider the primed axis rotated relative to the unprimed by an angle* ϕ *in the counterclockwise direction.*

(i) Derive the relation

$$A_x = A_x^{'} \cos\phi - A_y^{'} \sin\phi$$

$$A_y = A_y^{'} \cos\phi + A_x^{'} \sin\phi$$

that expresses unprimed components in terms of primed components of a vector $\vec{A}$ *using class notes if needed to get started.*

First note (by drawing a figure) that the rotated unit vectors are related to the old ones as follows

$$\mathbf{i}' = \mathbf{i}\cos\phi + \mathbf{j}\sin\phi$$

$$\mathbf{j}' = \mathbf{j}\cos\phi - \mathbf{i}\sin\phi$$

Now we have

$$\begin{aligned}
\mathbf{A} &= A_x^{'}\mathbf{i}' + A_y^{'}\mathbf{j}' && (1)\\
&= A_x^{'}(\mathbf{i}\cos\phi + \mathbf{j}\sin\phi) + A_y^{'}(\mathbf{j}\cos\phi - \mathbf{i}\sin\phi) && (2)\\
&= \mathbf{i}(A_x^{'}\cos\phi - A_y^{'}\sin\phi) + \mathbf{j}(A_y^{'}\cos\phi + A_x^{'}\sin\phi) && (3)
\end{aligned}$$

The coefficients of $\mathbf{i}$ and $\mathbf{j}$, are by definition A_x and A_y, yielding the desired result.

(ii) Invert these relations to express the primed components in terms of unprimed components. In doing this remember that the sines and cosines are constants and that we should treat $A_x^{'}$ *and* $A_y^{'}$ *as unknowns written in terms of knowns* A_x *and* A_y. *(Thus multiply one equation by something, another by something else, add and subtract etc to isolate the unknowns. Use simple trig identities)*

To find A'_x, we can multiply the equation for A_x by $\cos\phi$, the equation for A_y by $\sin\phi$ to get

$$\begin{aligned} A_x\cos\phi &= A'_x\cos^2\phi - A'_y\sin\phi\cos\phi && (4)\\ A_y\sin\phi &= A'_y\sin\phi\cos\phi + A'_x\sin^2\phi. && (5)\end{aligned}$$

Adding the two equations gives

$$A_x\cos\phi + A_y\sin\phi = A'_x(\cos^2\phi+\sin^2\phi) = A'_x,$$

because the terms containing A'_y cancel. Similarly, to find A'_y we multiply the equation for A_x by $\sin\phi$, the equation for A_y by $\cos\phi$ to get

$$\begin{aligned} A_x\sin\phi &= A'_x\sin\phi\cos\phi - A'_y\sin^2\phi && (6)\\ A_y\cos\phi &= A'_y\cos\phi^2 + A'_x\sin\phi\cos\phi. && (7)\end{aligned}$$

Subtracting the first equation from the second gives

$$A_y\cos\phi - A_x\sin\phi = A'_y(\cos\phi^2+\sin^2\phi) = A'_y.$$

Thus

$$\begin{aligned} A'_x &= A_x\cos\phi + A_y\sin\phi && (8)\\ A'_y &= A_y\cos\phi - A_x\sin\phi. && (9)\end{aligned}$$

(iii) *Argue why one could have obtained this result easily by reversing ϕ.* If I go from my axis to yours by a rotation ϕ clearly we go the other way with an angle $-\phi$.

(iv) *Consider a specific case $A_x = 1, A_y = 1, \phi = \pi/4$. What do you expect for A'_x and A'_y based on a sketch? Confirm by explicit calculation.*

We expect that since $\mathbf{A}$ is at an angle $\pi/4$ its entire length of $\sqrt{1^1+1^2} = \sqrt{2}$ will be along x' axis. Thus we should get $A'_x = \sqrt{2}$ and $A'_y = 0$. We find

$$A'_x = 1\cdot\cos\frac{\pi}{4} + 1\cdot\sin\frac{\pi}{4} = \sqrt{2}$$

$$A'_y = -1\cdot\sin\frac{\pi}{4} + 1\cdot\cos\frac{\pi}{4}4 = 0.$$

(v) *Verify that the length squared of $\vec{A}$ comes out same in both systems.*

$$\begin{aligned} (A'_x)^2+(A'_y)^2 &= (A_x\cos\phi + A_y\sin\phi)^2 + (A_y\cos\phi - A_x\sin\phi)^2 && (10)\\ &= A_x^2\cos^2\phi + 2A_xA_y\cos\phi\sin\phi + A_y^2\sin^2\phi \\ &\quad +A_y\cos^2\phi - 2A_xA_y\cos\phi\sin\phi + A_x^2\sin^2\phi && (11)\\ &= A_x^2(\cos^2\phi+\sin^2\phi) + A_y^2(\cos^2\phi+\sin^2\phi) && (12)\\ &= A_x^2 + A_y^2. && (13)\end{aligned}$$

(vi) *Consider another vector $\vec{B}$ and its components in the two frames. Show that*

$$A_x B_x + A_y B_y = A'_x B'_x + A'_y B'_y.$$

We will understand this invariance later.

We can write

$$\begin{aligned} B'_x &= B_x \cos\phi + B_y \sin\phi && (14) \\ B'_y &= B_y \cos\phi - B_x \sin\phi, && (15) \end{aligned}$$

so that (at this stage the algebra should be obvious, so we will skip a few steps)

$$\begin{aligned} A'_x B'_x + A'_y B'_y &= (A_x \cos\phi + A_y \sin\phi)(B_x \cos\phi + B_y \sin\phi) \\ &\quad +(A_y \cos\phi - A_x \sin\phi)(B_y \cos\phi - B_x \sin\phi) && (16) \\ &= A_x B_x (\cos^2\phi + \sin^2\phi) + A_y B_y (\cos^2\phi + \sin^2\phi) && (17) \\ &= A_x B_x + A_y B_y. && (18) \end{aligned}$$

4. *A particle is located at* $\mathbf{r}(t) = 14t\mathbf{i} + 6t^2\mathbf{j}$. *Find its position, velocity and acceleration at* $t = 2$ *s.*

$$\mathbf{r}(2) = 28\mathbf{i} + 24\mathbf{j}$$

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = 14\mathbf{i} + 12t\mathbf{j}$$

$$\mathbf{v}(2) = 14\mathbf{i} + 24\mathbf{j}$$

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = 12\mathbf{j}$$

5. *At a wedding the 2m tall bride throws her bouquet with a velocity* $v_0 = 25m/s$ *at an angle* 37^0 *above the horizontal. It is caught by a friend of height 1.5m. How long is the bouquet in flight and how far did it go horizontally? What was its maximum height above the ground?*

Stripped of all sentiment, this problem just involves a projectile with $y_0 = 2m$, $v_{ox} = 25\cos 37$, $v_{oy} = 25\sin 37$ and $a = -g = -9.8m/s^2$. Plug them into the equations for the two coordinates:

$$x(t) = v_{ox}t \qquad y(t) = y_0 + v_{oy}t - \frac{1}{2}gt^2$$

When the bouquet is caught we have $y(t) = 1.5$. Solve the above quadratic equation for t and plug into the one for x to see how far it flew. Next use

$$v_y(t) = \frac{dy}{dt} = v_{oy} - gt$$

to see when v_y vanishes. This gives the time at greatest height. Find the corresponding y for the height.

6.*Estimate the acceleration of the moon towards the earth given it orbits it once in 28 days at a radius of about a quarter of a million miles. (I know the units are funny and numbers* *are approximate. This problem tests your ability to give a quick and decent estimate, say to 10 percent .)*

Let the moon orbit the earth in time T at a radius R. Its speed is then $v = 2\pi R/T$ and its acceleration towards the earth is

$$a = \frac{v^2}{R} = \frac{4\pi^2 R}{T^2}.$$

Put in the given numbers to find an answer which is roughly 1/3600 times g. This will be significant when we come to Newton's Law of Gravity.

7. *A jet pilot diving vertically down at 600 km/hr wants to make a quarter turn without experiencing an acceleration bigger than* $5g$. *At what minimum height must the turn begin? Assume that the speed is constant and that after the quarter turn the plane, moving horizontally, is at ground level.*

The radius R of the quarter turn (which is also the height at which it must begin) determines

$$a = \frac{v^2}{R}$$

For this to be less than $5g$ we need

$$R \geq= \frac{v^2}{5g} = \frac{\left(\frac{600\cdot 1000}{3600}\right)^2}{5\cdot 9.8}.$$

Now you do the numbers!

8. *Here is problem designed so people in the life sciences will feel physics is relevant to them. A monkey is hanging from a height h and a person d meters away from the tree and on the ground, wants to zap it (in today's version with a tranquilizer gun and in the original version, a hunting rifle). He aims straight at the monkey and fires. This would of course work in the absence of gravity but show that it will work even in its presence provided the initial speed obeys* $v_0 > \sqrt{(d^2 + h^2)g/2h}$ *What does this requirement ensure? Given that this will also work if a pulse of laser light is used, what do you learn about light in a gravitational field?*
People interested in other areas can replace monkey by suitable object, e.g., hard drive or a copy of Kant's Critique of Pure Reason.

Let (x, y) and (X, Y) be vertical coordinates of monkey and bullet respectively. Choose the origin to be the location of person and let $(x = dy = h)$ be monkey's initial location. Draw a figure if you want.

We have

$$x(t) = d \quad \text{monkey just falls down at fixed x} \tag{19}$$

$$y(t) = h - \frac{1}{2}gt^2 \tag{20}$$

$$X(t) = v_0 \cos\theta \cdot t \tag{21}$$

$$Y(t) = v_0 \sin\theta \cdot t - \frac{1}{2}gt^2 \tag{22}$$

where $tan\theta = h/d$ ensures person aims at monkey.

The bullet hits the tree at time t^* where

$$X(t) = v_0 \cos\theta \cdot t^* = d \qquad \rightarrow t^* = \frac{d}{v_0 \cos\theta}$$

At this point its height is

$$Y(t^*) = v_0 \sin\theta \cdot t^* - \frac{1}{2} g t^{*2}$$

while the monkey is at a height

$$y(t^*) = h - \frac{1}{2} g t^{*2}.$$

Let us see if these two are equal. Canceling $\frac{1}{2} g t^{*2}$ from both sides we ask if

$$h = v_0 \sin\theta \cdot \frac{d}{v_0 \cos\theta}$$

and find that this indeed so given $\tan\theta = h/d$.

The condition $v_0 > \sqrt{(d^2 + h^2) g / 2h}$ ensures that the time it takes bullet to reach tree (called t^* above) is less than time it takes monkey to hit the ground ($\sqrt{2h/g}$).

If this works with a light pulse it means light too bends in a gravitational field. And it does, but by a different amount from what you get above to effects of General Relativity.