Problem Set IX Solutions
Fall 2006 Physics 200a

1. **Write the equation for a wave moving along +*x* with amplitude .4*m*, speed *6m/s* and frequency 17*Hz*. If these are waves on a string with mass per unit length μ = *.02kg/m*, what is the *u*, the energy per unit length? What is the power being fed into the vibrating string?**

   Equation (16-2) gives us the general equation for a moving wave:
   $$\psi(x,t) = A \cos(2\pi x/\lambda \pm 2\pi ft)$$
   We will take the – sign since we want a wave moving along the +*x* direction. But, we were given the wave velocity and frequency, not wavelength, so we must use Equation (16-1):
   $$v = \lambda f$$
   Solving this for λ and then substituting into the chosen form of Equation (16-2) gives:
   $$\psi(x,t) = A \cos(2\pi xf/v - 2\pi ft)$$
   $\Rightarrow\ \psi(x,t) = (0.4m) \cos[2\pi(17Hz)x/(6m/s) - 2\pi(17Hz)t]$
   $\Rightarrow\ \psi(x,t) = (0.4m) \cos[(17\pi/3\ m^{-1})x - (34\pi\ Hz)t]$

   From the lecture notes, we know that the energy per unit length for a sinusoidal wave on a string is:
   $$u = \tfrac{1}{2}\mu A^2\omega^2$$
   This is given in angular frequency, but we only have the frequency. Fortunately, Equation (15-7) reminds us:
   $$f = \omega / 2\pi \text{ or } 2\pi f = \omega$$
   Which gives the final result that:
   $$u = 2\pi^2\mu A^2 f^2$$
   $\Rightarrow u = 2\pi^2(.02kg/m)(0.4m)^2(17Hz)^2 \approx 18.3\ J/m$

   The power that must be fed into the string varies with what part of wave being produced, but the average power is given by Equation (16-8):
   $$\langle P\rangle = \tfrac{1}{2}\mu\omega^2A^2v$$
   This is again given in angular frequency when we only have the frequency. Again making use of Equation (15-7) lets us say that the average power is:
   $$\langle P\rangle = \tfrac{1}{2}\mu(2\pi f)^2A^2v = 2\pi^2\mu f^2A^2v$$
   $\Rightarrow \langle P\rangle = 2\pi^2(0.02kg/m)(17Hz)^2(0.4m)^2(6m/s) \approx 109.5W$

2. **The speed of sound in water and air is 1450*m/s* and 330*m/s* respectively. Sound from an explosion on the surface of a lake first reaches me when my head is underwater and 5*s* later when my head is above the water. How far away was the explosion?**

   From equation (2-1) we know that $t = D / v$ where t is time spent moving, D is the total distance moved and v is the velocity when moving. From the problem we can construct the following two equations:

   (1) $t_{air} = D / v_{air}$

(2) $t_{water} = D / v_{water}$

But we also know that $t_{air} = t_{water} + \Delta t$ where Δt is $5s$, the difference in travel times. So, we can rewrite our equations as:

(1*) $t_{water} + \Delta t = D / v_{air}$ or $t_{water} = (D / v_{air}) - \Delta t$

(2*) $t_{water} = D / v_{water}$

If we then combine the second form of (1*) and (2*) we get:

$$(D / v_{air}) - \Delta t = D / v_{water}$$

$\Rightarrow D \bullet v_{water} - \Delta t \bullet v_{air} \bullet v_{water} = D \bullet v_{air}$

$\Rightarrow D \bullet (v_{water} - v_{air}) = \Delta t \bullet v_{air} \bullet v_{water}$

$\Rightarrow D = (\Delta t \bullet v_{air} \bullet v_{water}) / (v_{water} - v_{air})$

Evaluating with the given numeric values gives:

$$D = (5s)(330m/s)(1450m/s) / (1450m/s - 330m/s) = 2136.16m \approx 2140m$$

3. **A block of mass M sits on a frictionless inclined plane of angle $\alpha = \pi/4$ as in Figure (1). It is connected by a wire of linear mass density $\mu = .03kg/m$ that goes over a pulley that supports mass m. Both masses are at rest. If transverse waves travel at $v = 80m/s$ in the wire find M and m, using symbols till the end. Ignore the mass of the string in computing the tension on the string; use it just to find the velocity of waves.**

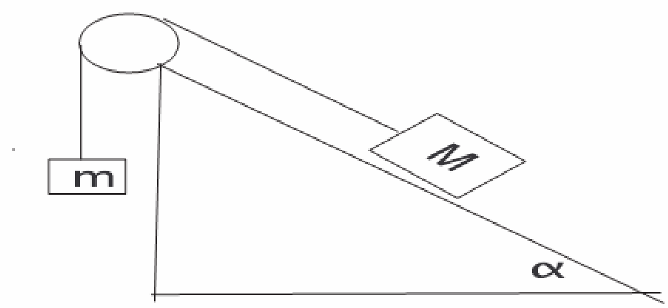
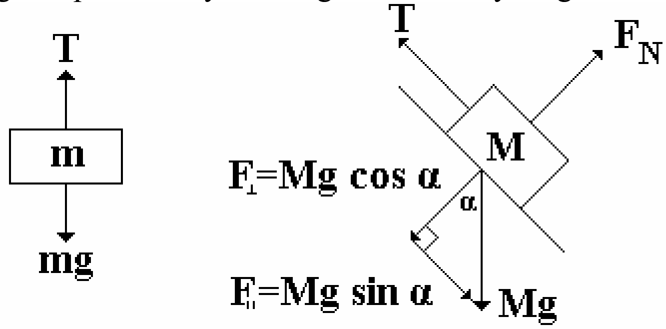


FIG. 1. The masses are at rest and the wire carries transverse waves at $v = 80m/s$.

Let us begin solving this problem by drawing the free body diagrams for both masses.



Since we know that both masses are at rest that means that the forces on each mass must balance. Focusing our attention on the tension in the wire gives us these two equations for the magnitudes of the forces:

$$T = mg$$
$$T = Mg \sin \alpha$$

From which we can easily solve for the masses:

$$(1)\ m = T/g$$
$$(2)\ M = T/(g \sin \alpha)$$

The statement of the problem also gave us the wave velocity in the wire which we can make use of through Equation (16-7):

$$v = (T/\mu)^{1/2}$$

which prompts us to solve for the tension.

$$T = \mu v^2$$

If we combine this result with equations (1) and (2) we have our solutions for M and m:

$$m = \mu v^2/g$$

$\Rightarrow m = (0.03kg/m)(80m/s)^2 \,/\, (9.8m/s^2) \approx 19.6kg$

$$M = \mu v^2/(g \sin \alpha)$$

$\Rightarrow M = (0.03kg/m)(80m/s)^2 \,/\, [(9.8m/s^2)(\sin \pi/4)] \approx 27.7kg$

4. What is the ratio of sound intensities for which the difference is 1*dB*?

Equation (17-4) defines the sound intensity level for us in terms of intensities like so:

$$\beta = 10 \log(I \,/\, I_o)$$

where I_o is the reference level chosen as the threshold of hearing at 1*kHz*. If we have two sounds and want them to have a sound intensity level difference of 1dB then we can write:

$$\beta_1 - \beta_2 = 10 \log(I_1 \,/\, I_o) - 10 \log(I_2 \,/\, I_o) = 1\ dB$$

with the definition that $\beta_1 > \beta_2$. It follows that:

$$\begin{aligned}\beta_1 - \beta_2 &= 10 \log(I_1 \,/\, I_o) - 10 \log(I_2 \,/\, I_o)\\ &= 10\,[\log(I_1 \,/\, I_o) - \log(I_2 \,/\, I_o)]\\ &= 10 \log\,([I_1 \,/\, I_o]/[I_2 \,/\, I_o])\\ &= 10 \log\,(I_1 \,/\, I_2) = 1\ dB\end{aligned}$$

But we want the ratio of intensities, so we can isolate the log term in the last equation and then exponentiate.

$$10 \log\,(I_1 \,/\, I_2) = 1\ dB$$

$\Rightarrow \log\,(I_1 \,/\, I_2) = 0.1\ dB$

$\Rightarrow 10^{\log\,(I_1 \,/\, I_2)} = 10^{0.1\ dB}$

$\Rightarrow I_1 \,/\, I_2 = 10^{0.1} \approx 1.26$

5. A source of sound radiates uniformly in all directions. Along a radial line from the source locate two points separated by 2*m* such that the intensity at the nearer point is 4*dB* above that of the more distant point.

Equation (16-9) tells us that the dependence of intensity on total emitted power and radial distance to the source for spherical waves takes the form:

$$I = P / (4\pi r^2)$$

Now, we want to consider only two points along the same radial line such that they are 2*m* apart and that the intensity at the point closer to the sound source is 4*dB* higher than the other. The first requirement can be satisified by the equation:

$$(1) \qquad r_2 = r_1 + 2m$$

And the second we obtain by following the same logic as in Problem 4., resulting in:

$$(2) \qquad I_1 / I_2 = 10^{0.4\ dB}$$

Combining (2) with (16-9) yields:

$$[P / (4\ \pi r_1^2)] / [P / (4\ \pi r_2^2)] = 10^{0.4}$$

$\Rightarrow r_2^2 / r_1^2 = 10^{0.4}$

$\Rightarrow (r_2 / r_1)^2 = 10^{0.4}$

$\Rightarrow r_2 / r_1 = (10^{0.4})^{1/2} = 10^{0.2}$

$\Rightarrow r_2 = 10^{0.2}\ r_1$

We can now take advantage of (1) to get a solution.

$\Rightarrow\ r_1 + 2m = 10^{0.2}\ r_1$

$\Rightarrow\ 2m = 10^{0.2}\ r_1 - r_1 = (10^{0.2} - 1)\ r_1$

$\Rightarrow\ r_1 = 2m / (10^{0.2} - 1) \approx 3.42m$

So, the two points are located at 3.42*m* and 5.42*m* from the source.

6. I place a massless speaker emitting sound at 600*Hz* on top of a mass connected to a spring. I now set the mass-spring system in an oscillatory state, vibrating horizontally at 4*Hz* with amplitude *A*. Given v_{sound} = 330*m/s*, and that the difference between the highest and lowest frequencies I hear is 2*Hz*, what is *A*? If I now turn on another identical sound source, what will be the largest beat frequency I will hear?

Let us first consider the horizontally vibrating mass-spring system. This type of system is an example of simple harmonic motion, and we can find the time dependent velocity from Equation (15-10)

$$v(t) = -A\omega \sin(\omega t)$$

Since all we care about for this problem are the maximum and minimum velocities reached we don't have to worry about the issue of when we define t=0. All we need to do is note that the maximum and minimum values of sin x are $\pm$ 1. So we see that:

$$v_{max} = A\omega$$

However the problem gave us the frequency, not the angular frequency, but Equation (15-7) reminds us:

$$f_{osc} = \omega / 2\pi \text{ or } 2\pi f_{osc} = \omega$$

So we can simplify to:

$$v_{max} = 2\pi A f_{osc}$$

That we can hear a difference in frequencies means that a Doppler shift is occurring. The equation for a Doppler shift due to a moving source is given by Equation (17-10):

$$f' = f / (1 \pm u / v_{sound})$$

where f' is the observed frequency by the observer at rest, f is the frequency that the speaker is emitting, and u is the velocity that the speaker is moving at relative to the observer. So, since we know what the difference is between the highest and lowest frequencies heard, we can write:

$$f'_{\text{high}} - f'_{\text{low}} = (f / (1 - v_{max}/ v_{sound})) - (f / (1 + v_{max} / v_{sound})) = 2\ Hz$$

$\Rightarrow f \bullet \{ (1 + v_{max} / v_{sound}) - (1 - v_{max}/ v_{sound})\} = (2\ Hz) \bullet (1 + v_{max} / v_{sound}) \bullet (1 - v_{max}/ v_{sound})$

$\Rightarrow f \bullet (2v_{max} / v_{sound}) = (2\ Hz) \bullet \{ 1 - (v_{max} / v_{sound})^2\}$

$\Rightarrow (v_{max} / v_{sound})^2 + (f / 1Hz) \bullet (v_{max} / v_{sound}) - 1 = 0$

If we introduce the change of variables $x = v_{max} / v_{sound}$ and let f now be unit-less, we get the equation:

$$x^2 + fx - 1 = 0$$

Which has only one positive solution:

$$x = ½(f^2 + 4)^{½} - f/2$$

So now we know that:

$$v_{max}/ v_{sound} = ½(f^2 + 4)^{½} - f/2$$

$\Rightarrow 2\pi A f_{osc} = (v_{sound}/2) \bullet \{(f^2 + 4)^{½} - f\}$

$\Rightarrow A = v_{sound} \bullet \{(f^2 + 4)^{½} - f\} / (4\pi f_{osc}) = (330m/s)\{(600^2+4)^{½} - 600\} / (4\pi \bullet 4Hz) \approx 0.022m$

If you now turn on another source of sound you will observer a beat pattern that has a frequency equal to the magnitude of the difference between the two sources. The highest and lowest frequencies generated by the moving source are:

$f'_{\text{high}} = f / (1 - v_{max}/ v_{sound}) = f / (1 - 2\pi A f_{osc} / v_{sound})$
$\quad = 600Hz / (1 - 2\pi(0.022m)(4Hz)/330\ m/s) \approx 601\ Hz$

$f'_{\text{low}} = f / (1 + v_{max} / v_{sound}) = f / (1 + 2\pi A f_{osc} / v_{sound})$
$\quad = 600Hz / (1 + 2\pi(0.022m)(4Hz)/330\ m/s) \approx 599\ Hz$

So, once the new 600*Hz* source is turned on the largest beat frequency that you can hear is ~1*Hz*.

7. **A mass *M* hangs vertically at the end of a cable of mass *m* and length *L*.**
(i) How long will a transverse pulse take to travel from bottom to top if you ignore *m*, the cable mass?
(ii) Now repeat, including *m* and remembering that the velocity of the signal varies with the distance from the bottom end. Show that the answer reduces to part (i) if you set *m*=0.

(i) The way in which we ignore the cable mass is to assume that the tension in the cable is constant and due only to the mass hanging at the end. Thus the tension must exactly cancel the force of gravity on the mass and we are left with:

$$T = Mg$$

To find the velocity with which the wave will travel we can use Equation (16-7):

$$v = (T/\mu)^{½}$$

where μ is the linear mass density of the cable; here $\mu = m/L$. From equation (2-1) we know that $t = L / v$. Combining these equations gives:

$$t = L / [Mg /(m/L)]^{½} = (mL / Mg)^{½}$$

(ii) This time we won't ignore the cable's mass when calculating the tension. At any given point along the cable the tension will have to balance both the mass at the end of the cable and any mass from the portion of the cable below it. So we should say that:

$$T = (M + xm/L)g$$

Where x is the position along the cable setting x=0 at the point that the mass M is connected. Again making use of Equation (16-7) we get that:

$$v = [(M + xm/L)g \,/\, \mu]^{½}$$

(We won't immediately write μ in terms of m and L because that would confuse things a bit when we try to show that part (i) is a limiting case for this solution.) We can see here that the velocity is explicitly dependent on the position and so the simple method used in part (i) won't work. Instead we make use of the fact that v = dx/dt and construct a differential equation.

$$\mathrm{d}x/\mathrm{d}t = [(M + xm/L)g \,/\, \mu]^{½}$$
$$x(0) = 0$$

The initial condition comes from the fact that the pulse is traveling from the bottom to the top of the cable. This differential equation can be solved via a separation of variables as follows:

$$\mathrm{d}x/\mathrm{d}t = [(M + xm/\mathrm{L})\mathrm{g} \,/\, \mu]^{½}$$

$\Rightarrow (M + xm/L)^{-½}\mathrm{d}x = (g \,/\, \mu)^{½}\,\mathrm{d}t$
$\Rightarrow \int (M + xm/L)^{-½}\mathrm{d}x = \int (g \,/\, \mu)^{½}\,\mathrm{d}t$
$\Rightarrow (2L/m)(M + xm/L)^{½} = t\bullet(g \,/\, \mu)^{½} + C\,*$
$\Rightarrow (M + xm/L)^{½} = t\bullet(m/2L)(g \,/\, \mu)^{½} + mC/(2L)$
$\Rightarrow M + xm/L = m^2gt^2/(4L^2\mu) + m^2tC(g/\mu)^{½}/(2L^2) + m^2C^2/(4L^2)$
$\Rightarrow x = mgt^2/(4L\mu) + t\bullet mC(g/\mu)^{½}/(2L) + mC^2/(4L) - LM/m$
(* C is the overall constant from the two integrations.)

To remove the constant of integration we put the initial condition that $x(0) = 0$ to use.
$\Rightarrow 0 = mC^2/(4L) - LM/m$
$\Rightarrow C^2 = 4L^2M/m^2$
$\Rightarrow C = \pm 2LM^{½}/m$

$\Rightarrow x = mgt^2/(4L\,\mu) \pm t\bullet(Mg/\mu)^{½}$

Here we can kill two birds with one stone. We need to show that in the limit that m=0 this solution is the same as in part (i) and we need to choose which of the two solutions is correct. First let us take the proper limit, removing the first term and look at the condition that the wave is done traveling, i.e. x=L. This leaves:

$$L = \pm\, t\bullet(Mg/\mu)^{½}$$

Solving for t leaves us with:

$$t = \pm\,(L^2\mu/Mg)^{½}$$

We can now substitute in $\mu = m/L$ to arrive at:

$$t = \pm\,(mL \,/\, Mg)^{½}$$

This shows that we get the correct answer in the m=0 limit. Additionally, we now know that only the + solution is of interest, which looking back is obvious since only that solution guarantees positive values for x for any positive value of t.

Going back to the full solution for x and making the subsitituions that $x=L$ and $\mu = m/L$ we get the following quadratic equation for x:

$$L = gt^2/4 + t\bullet(MgL/m)^{1/2}$$

Which has only one positive solution, namely:

$$t = 2\{[L(m+M) / mg]^{1/2} - (ML / mg)^{1/2}\}$$

8. **Two speakers emitting sound at 550*Hz* are 1.5*m* apart. The first destructive interference takes place 4*m* to the right and 0.8*m* above the line of symmetry, as in Figure (2). What is the velocity of sound? Do this using the Pythagoras theorem to calculate distances exactly and compare to the small-angle approximation $d \sin\theta = (n + 1/2)\lambda$.**

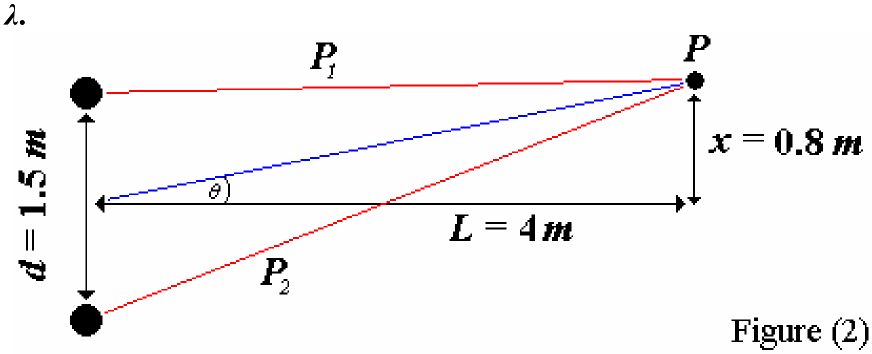


Figure (2)

To make things easier to follow let's draw the figure a little closer to scale and give all of the relevant distances in the problems names as in the modified version of Figure (2). We are told that Point P is the location of the first point of destructive interference. Destructive interference occurs whenever the sound waves from the two speakers are 180° out of phase. This first occurs when the path length difference is one half of a wavelength. First we write out the path lengths P_1 and P_2 in terms of known distances:

$$P_1 = [(d/2 - x)^2 + L^2]^{1/2}$$
$$P_2 = [(d/2 + x)^2 + L^2]^{1/2}$$

(It is worth noting that the original drawing in the problem set is not to scale and point P is actually "above" the top speaker.) So the condition of the first destructive interference is:

$$P_2 - P_1 = \lambda/2$$

However, we want the velocity of the sound, not the wavelength, so we will use Equation (16-1):

$$v = \lambda f$$

This will allow us to write out an expression for the velocity of sound only in terms of known quantities.

$$v = 2f\bullet(P_2 - P_1) = 2f\bullet\{[(d/2 + x)^2 + L^2]^{1/2} - [(d/2 - x)^2 + L^2]^{1/2}\}$$
$$\Rightarrow v = 2(550Hz)\{[(1/2\bullet 1.5m + 0.8m)^2 + (4m)^2]^{1/2} - [(1/2\bullet 1.5m - 0.8m)^2 + (4m)^2]^{1/2}\}$$
$$\approx 318.45 m/s$$

If we were to solve this with the small angle formula:

$$d \sin\theta = (n + 1/2)\lambda$$

for the first destructive interference (n=0) we should again make use of Equation (16-1) and solve for v getting:

$$v = 2fd \sin\theta$$

But from simple geometric arguments we see that:

$$\sin\theta = x/(L^2 + x^2)^{1/2} = [1+(L/x)^2]^{-1/2}$$

$\Rightarrow v = 2fd\bullet[1+(L/x)^2]^{-1/2}$

$\Rightarrow v = 2(550Hz)(1.5m)[1 + (4m / 0.8m)^2]^{-1/2} \approx 323.59m/s$

So the small angle approximation gives an answer that differs by only about 5.14m/s or about 1.5% from the exact answer.

9. The ear canal is about 3*cm* long and can be viewed as a tube open at one end and closed at the other. Relate this to the fact that we seem to hear best at around 3000*Hz*.

Figure 17-16 has a good visualization of a tube with one open end. The first image of 17-16(a) is the first resonant mode in this set up. That is, a node at the closed end and an anti-node at the open end. Just as with the string (see Figure 17-13) this means that the length of the tube must be one quarter of a wavelength. We will take the speed of sound to be 330 m/s as was given in Problem 6. Equation (16-1) gives us:

$$v = \lambda f$$

So if we solve for f we should get the first resonant frequency in the ear.

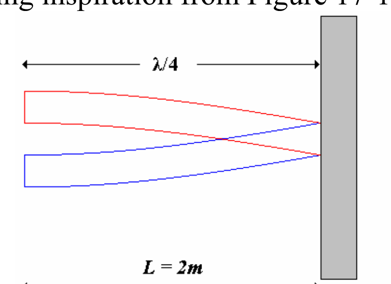
$$f = v/\lambda = v/4l$$

$$f = (330m/s)/(4\bullet 0.03m) = 2750Hz \approx 3000Hz$$

Unsurprisingly, we seem to hear best around the resonant frequency of our ears.

10. Longitudinal waves on a metal rod travel at 3450*m/s*. Find two of the lowest standing wave frequencies on a rod of length 2*m* clamped at one end and free at the other. Draw figures. Repeat if rod is clamped at both ends.

Starting with the lowest standing wave frequency for the rod clamped on one end, we first draw a picture. Since one end is fixed it must be a node of the wave, and the open end must be an antinode. For the lowest standing wave frequency we want no other nodes or antinodes. Drawing inspiration from Figure 17-13:



To find the frequency of this standing wave, we use Equation (16-1):

$$v = \lambda f$$

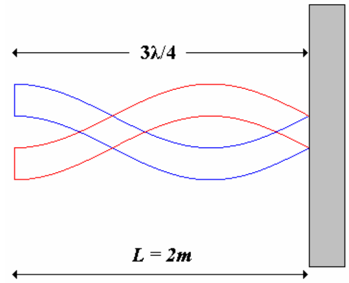
From our forced locations of nodes and antinodes in the drawing we also know that:

$$L = \lambda/4 \text{ or } \lambda = 4L$$

Substituting and solving for f we conclude that the frequency of the standing wave is:

$$f = v/(4L) = (3450m/s) / (4\bullet 2m) = 431.25Hz$$

The next lowest frequency standing wave must still have a node at the fixed end and an antinode at the open end. To make it distinct from the last wave and still fit this constraint we must add an internal antinode and node:



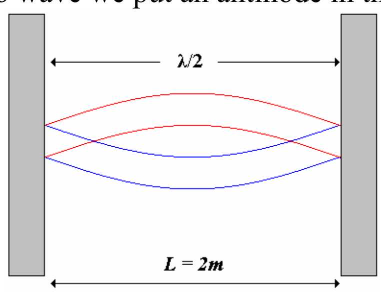
Counting the number of nodes and antinodes in the drawing we see that here:

$$L = 3\lambda/4 \text{ or } \lambda = 4L/3$$

Substituting and solving for f we conclude that the frequency of the standing wave is:

$$f = 3v/(4L) = (3 \bullet 3450m/s) / (4 \bullet 2m) = 1293.75Hz$$

For the case that the rod is fixed at both ends, the wave must have a node at both ends. To avoid the trivial case of no wave we put an antinode in the middle to get:



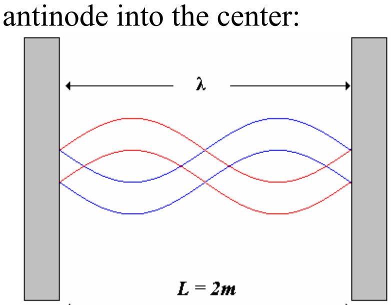
This is the rather clearly half of a wave so that we see:

$$L = \lambda/2 \text{ or } \lambda = 2L$$

Substituting and solving for f we conclude that the frequency of the standing wave is:

$$f = v/(2L) = (3450m/s) / (2 \bullet 2m) = 862.5Hz$$

The next lowest frequency standing wave when both ends are clamped must still have nodes at both ends. The smallest change that we make to the wave to keep this condition is adding an extra node and antinode into the center:



We can see that this is a full wave, so we get:

$$L = \lambda$$

Substituting and solving for f we conclude that the frequency of the standing wave is:

$$f = v/(L) = (\ 3450m/s)\ /\ (2m) = 1725Hz$$

11. How far apart are the nodes on a string 80cm long vibrating at 1600*Hz* assuming a wave velocity on the string of 320*m/s*?

Since we know both the wave velocity and the frequency, let's use Equation (16-1) to find the wavelength:

$$v = \lambda f$$

$$\Rightarrow \lambda = v/f = (320m/s)/(1600Hz) = 0.2m = 20cm$$

Since we know that nodes must be spaced every half wavelength (including the ends of our string), we should have 8 nodes on the string spaced every 10*cm*.