Physics 200
Problem Set 6
Solution

1. (i) What is the moment of inertia I_{CM} of a propeller with three blades (treated as rods) of mass m, length L, at 120° relative to each other? (ii) If a torque τ acts on this how long will it take to reach an angular velocity ω? (iii) How many revolutions will it have made before reaching this ω? (iv) Get the numerical answers if $L = 1.25$ m, $m = 12$ kg, $\tau = 3000$ N $\cdot$ m, $\omega = 2000$ rad/s.

   Answer:

   (i) We know (if not, look on p. 296) that the moment of inertia of a single rod rotating around its end is $\frac{1}{3}mL^2$. It's not hard to convince oneself that if there are three of them rotating around the same axis and in the same plane, the moment of inertia is just three times this, $I_{\text{CM}} = mL^2$.

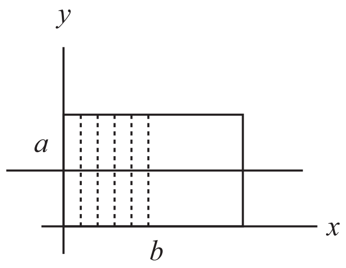
   (ii) Since $\omega = \alpha t$ and $\tau = I_{\text{CM}}\alpha$,
   $$t = \frac{I_{\text{CM}}\omega}{\tau} = \frac{mL^2\omega}{\tau}.$$

   (iii) From our knowledge of constant acceleration problems,
   $$\omega^2 = 2\alpha\theta \implies \theta = \frac{\omega^2}{2\alpha} = \frac{I_{\text{CM}}\omega^2}{2\tau} = \frac{mL^2\omega^2}{2\tau}.$$
   The number of revolutions it made is
   $$N = \frac{\theta}{2\pi} = \frac{mL^2\omega^2}{4\pi\tau}.$$

   (iv)
   $$\begin{aligned} I_{\text{CM}} &= (12 \text{ kg})(1.25 \text{ m})^2 = 19 \text{ kg} \cdot \text{m}^2, \\ t &= \frac{(12 \text{ kg})(1.25 \text{ m})^2(2000 \text{ s}^{-1})}{2(3000 \text{ N} \cdot \text{m})} = 13 \text{ s}, \\ N &= \frac{(12 \text{ kg})(1.25 \text{ m})^2(2000 \text{ s}^{-1})^2}{4\pi(3000 \text{ N} \cdot \text{m})} = 2000. \end{aligned}$$

2. Consider I for a rectangle of sides a (along the y-axis) and b (along the x-axis) about the two symmetry axes. (Rotate the rectangle about one of these axes and think of it as composed of rods.) Show that about the axis parallel to x, $I = \frac{1}{12}Ma^2$. Going back to the very definition of I, show that if this rectangle is spun around an axis through its CM and perpendicular to its area the moment of inertia will be $I = \frac{1}{12}M(a^2 + b^2)$.

   Answer: To find I_x, the moment of inertia about the symmetry axis parallel to the x-axis, we think of dividing the rectangle into many thin rods:

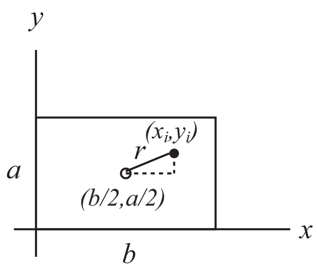


Since each "rod" has length a, it should be obvious that the sum of these contributions is simply $\frac{1}{12}Ma^2$, i.e., the same as if there was one rod of mass M rotating around the axis, but let's be more explicit. If we divide up the rectangle into n rods (where n is large so they really are rods, though in the end it doesn't matter) so that each rod has mass M/n, then with each rod contributing $\frac{1}{12}(\frac{M}{n})a^2$ and there being n rods, we get $I_x = \frac{1}{12}Ma^2$ as expected. The same logic of course applies to the symmetry axis parallel to the y-axis.

From the definition of the moment of inertia,

$$I = \sum_i r_i^2 \Delta m_i.$$

Notice from the picture that $r_i^2 = (x_i - b/2)^2 + (y_i - x/2)^2$, so we can write



$$I = \sum_i \left(x_i - \frac{b}{2}\right)^2 \Delta m_i + \sum_i \left(y_i - \frac{a}{2}\right)^2 \Delta m_i = I_y + I_x = I_x + I_y = \frac{1}{12}M(a^2 + b^2).$$

It's easy to see that this kind of argument will work for any *flat* object, as long as you choose your axes to be perpendicular to each other.

3. A 4.8 kg block is resting at the top of a 30° slope of height 1 m. It is attached to a cylindrical pulley of mass 1.7 kg and radius 8 cm by a massless string that unwinds as the block slides downhill. If the acceleration of the block is 1.9 m/s^2 what is μ_k? Find the velocity at the bottom of the slope using forces and torques. Repeat using energy ideas. See Fig. 1.

   Answer: Numbers are cumbersome, so let's start by letting m be the mass of the block, θ the angle, h the height, M the mass of the pulley, R its radius, and a the acceleration felt by the block. As usual, we decompose the forces on the block into those that are parallel to the slope and those that are perpendicular to the slope. In the perpendicular direction, the component of gravity in this direction $mg\cos\theta$ balances the normal force, so $N = mg\cos\theta$. In the parallel direction, we have

$$mg\sin\theta - T - f_k = ma$$

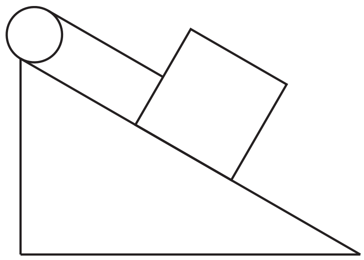


Figure 1: The block started at height h.

(we take downhill to be positive), where the kinetic friction term is given by $f_k = \mu_k N = \mu_k mg\cos\theta$. We now need the tension T, which we find as follows. The acceleration a that the block feels is converted to angular acceleration in the pulley, where $a = R\alpha$. But we also know that $\tau = RT = I\alpha$ where $I = \frac{1}{2}MR^2$ is the moment of inertia of the pulley. Therefore

$$T = \frac{I\alpha}{R} = \frac{Ia}{R^2} = \frac{1}{2}Ma,$$

and plugging into the above equation we get

$$f_k = mg\sin\theta - T - ma = mg\sin\theta - \left(\frac{M}{2} + m\right)a.$$

Then

$$\mu_k = \frac{f_k}{mg\cos\theta} = \frac{mg\sin\theta - (M/2+m)a}{mg\cos\theta}.$$

Before plugging in the numbers, note that we can make our lives just a little easier by rearranging a little bit:

$$\mu_k = \frac{\sin\theta - (1+M/2m)a/g}{\cos\theta} = \frac{\sin 30^\circ - [1 + ((1.7\text{ kg})/2(4.8\text{ kg}))(1.9\text{ m/s}^2)/(9.81\text{ m/s}^2)}{\cos 30^\circ} = 0.31.$$

Since the block travels a distance $d = h/\sin\theta$, the velocity at the bottom of the slope, starting from rest at the top, is

$$v^2 = 2ad = \frac{2ah}{\sin\theta} \implies v = \sqrt{\frac{2ah}{\sin\theta}} = \sqrt{\frac{2(1.9\text{ m/s}^2)(1\text{ m})}{\sin 30^\circ}} = 2.8\text{ m/s}.$$

The energy method is somewhat more involved. At the top, the energy of the block is purely potential energy, mgh. At the bottom, the potential energy is zero, the kinetic energy of the block is $T = \frac{1}{2}mv^2$, and the rotational kinetic energy of the pulley is

$$\frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2 = \frac{1}{4}Mv^2.$$

Some energy was also lost to friction, so we must take that into account. The work done by friction is

$$f_k d = \mu_k mg\cos\theta \cdot \frac{h}{\sin\theta} = \frac{\mu_k mgh}{\tan\theta}.$$

We can finally use the conservation of energy in the form

$$mgh = \frac{1}{2}mv^2 + \frac{1}{4}Mv^2 + \frac{\mu_k mgh}{\tan\theta},$$

which means

$$v = \sqrt{\frac{2gh(1 - \mu_k/\tan\theta)}{1 + M/2m}} = \sqrt{\frac{2(9.81\ \text{m/s}^2)(1\ \text{m})(1 - 0.31/\tan 30°)}{1 + (1.7\ \text{kg})/2(4.8\ \text{kg})}} = 2.8\ \text{m/s}.$$

As one would expect, the two answers agree.

4. Argue that that $\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = 0$. In three dimensions find the expression of $\mathbf{A} \times \mathbf{B}$ in terms of vector components and $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$.

   Answer: Since $\mathbf{A} \times \mathbf{B}$ is always perpendicular to both $\mathbf{A}$ and $\mathbf{B}$, the dot product with $\mathbf{A}$ must be zero. For the second part, we use the basic relations

   $$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0, \qquad \hat{\mathbf{i}} \times \hat{\mathbf{j}} = -\hat{\mathbf{j}} \times \hat{\mathbf{i}} = \hat{\mathbf{k}}, \qquad \hat{\mathbf{j}} \times \hat{\mathbf{k}} = -\hat{\mathbf{k}} \times \hat{\mathbf{j}} = \hat{\mathbf{i}}, \qquad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = -\hat{\mathbf{i}} \times \hat{\mathbf{k}} = \hat{\mathbf{j}}.$$

   Thus when we expand the cross product, we can immediately drop any term that contains the same unit vector twice:

   $$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}) \times (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}) \\ &= A_xB_y(\hat{\mathbf{i}} \times \hat{\mathbf{j}}) + A_xB_z(\hat{\mathbf{i}} \times \hat{\mathbf{k}}) + A_yB_x(\hat{\mathbf{j}} \times \hat{\mathbf{i}}) + A_yB_z(\hat{\mathbf{j}} \times \hat{\mathbf{k}}) + A_zB_x(\hat{\mathbf{k}} \times \hat{\mathbf{i}}) + A_zB_y(\hat{\mathbf{k}} \times \hat{\mathbf{j}}) \\ &= (A_yB_z - A_zB_y)\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\hat{\mathbf{k}}. \end{aligned}$$

   We can also check that

   $$\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = A_x(A_yB_z - A_zB_y) + A_y(A_zB_x - A_xB_z) + A_z(A_xB_y - A_yB_x) = 0,$$

   as we argued.

5. A disk of radius R and mass M is spinning at an angular velocity ω_0 rad/s. A non-rotating concentric disk of mass m and radius r drops on it from a negligible height and the two rotate together. (See Fig. 2). Find the final ω and fraction of initial kinetic energy left.

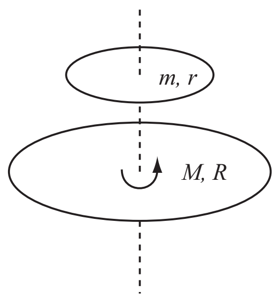


Figure 2: The upper disk, initially at rest, falls with negligible speed on the lower one which is spinning. Their centers coincide.

   Answer: Let $I_M = \frac{1}{2}MR^2$ be the moment of inertia of the disk of radius R, and $I_m = \frac{1}{2}mr^2$ the moment of inertia of the other disk. Initially the angular momentum of the system is $I_M\omega_0$, and when the two disks are rotating together the angular momentum is $(I_M + I_m)\omega$. Since angular momentum is conserved,

   $$I_M\omega_0 = (I_M + I_m)\omega \implies \omega = \frac{I_M\omega_0}{I_M + I_m} = \frac{\omega_0}{1 + mr^2/MR^2}.$$

Before the second disk is added, the rotational kinetic energy of the system is $\frac{1}{2}I_M\omega_0^2$, and afterward the kinetic energy is $\frac{1}{2}(I_M + I_m)\omega^2$. Therefore the fraction of the initial kinetic energy left is

$$\frac{\frac{1}{2}(I_M + I_m)\omega^2}{\frac{1}{2}I_M\omega_0^2} = \frac{I_M + I_m}{I_M}\left(\frac{\omega}{\omega_0}\right)^2 = \frac{I_M + I_m}{I_M}\left(\frac{I_M}{I_M + I_m}\right)^2 = \frac{I_M}{I_M + I_m} = \frac{1}{1 + mr^2/MR^2}.$$

Evidently the kinetic energy is conserved only if the initially rotating disk has infinite moment of inertia, or if the second disk has no moment of inertia.

6. A gyro consists of a solid disk of radius R mounted at one end of a shaft of zero mass and length l, the other end of which is on a pivot. The disk spins at ω rad/s and the gyro precesses at ω_p rad/s. What is l in terms of ω_p, ω, g and R? Give a number for l when $R = 6$ m, $\omega_p = 2.6$ rad/s, and the disc is spinning at 450 rpm?

   Answer: Note that the mass of the disk is not given in this problem, so we'll say the mass is M and see why it doesn't matter what the mass is. The magnitude of the angular momentum is $L = I\omega = \frac{1}{2}MR^2\omega$. Since gravity acts at distance l from the pivot, the torque produced by the force $\tau = Mgl$, and so the equation for a gyroscope tells us

   $$\omega_p = \frac{\tau}{L} = \frac{Mgl}{MR^2\omega/2} = \frac{2gl}{R^2\omega}.$$

   As we hoped, the mass has canceled from the equation. Solving for the length of the shaft gives

   $$l = \frac{R^2\omega\omega_p}{2g}.$$

   450 rpm is

   $$450 \text{ rpm} \cdot \frac{2\pi \text{ rad}}{1 \text{ rev}}\frac{1 \text{ min}}{60 \text{ s}} = 47 \text{ rad/s},$$

   so

   $$l = \frac{(6 \text{ m})^2(47 \text{ s}^{-1})(2.6 \text{ s}^{-1})}{2(9.81 \text{ m/s}^2)} = 220 \text{ m},$$

   which is quite a large number.

7. Two beads of mass m are free to slide on a rod of length l and mass M as in Fig. 3. Initially the beads are at the center and the rod is spinning freely (with no external torque) at ω_0 rad/s about a vertical axis through its center. Slowly the beads move radially out (at negligible velocity). (i) Find $\omega(r)$, the angular velocity when the beads are r m from the center. (ii) What is ω when they just fly tangentially off the rod? Argue that ω does not change hereafter. (Hint: Using the formula $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ for the angular momentum of beads show that their $\mathbf{L}$'s do not change even though $\mathbf{r}$'s do. Also are there any torques on the spinning rod after the beads detach?) (iii) Why was the force of friction between beads and rod unimportant in the preceding discussion?

   Answer:

   (i) Initially the moment of inertia of the system is due only to the rod, because $r = 0$ for the beads. So $I_0 = \frac{1}{12}Ml^2$, and the angular momentum is $L = I_0\omega_0$. If the beads are a distance r from the center, each one contributes mr^2 to the system's moment of inertia, so $I(r) = I_0 + 2mr^2$. Since angular momentum is conserved, $I_0\omega_0 = I\omega$, or

   $$\omega(r) = \frac{I_0\omega_0}{I} = \frac{I_0\omega_0}{I_0 + 2mr^2} = \frac{\omega_0}{1 + 2mr^2/I_0} = \frac{\omega_0}{1 + 24mr^2/Ml^2}.$$

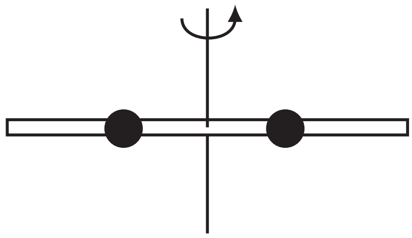
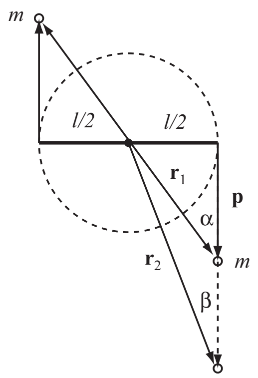


Figure 3: The point-like beads can slide along the rod which is spinning around a vertical axis through its CM. The beads start out at the center of the rod.

(ii) The beads fly off the rod when $r = l/2$, so at this time

$$\omega\left(\frac{l}{2}\right) = \frac{\omega_0}{1 + 6m/M}.$$

Once the beads fly off the rod, they continue moving in a straight line at constant velocity, so $\mathbf{p}$ is a constant. For each bead the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ points out of the page in the figure and has magnitude $|\mathbf{r}||\mathbf{p}|\sin\theta$. Since $|\mathbf{p}|$ stays constant, the angular momentum depends entirely on the quantity $|\mathbf{r}|\sin\theta$, and looking at the figure below where the position of one of the beads is indicated at two different times after flying off the rod, we see that $|\mathbf{r}_1|\sin\alpha = |\mathbf{r}_2|\sin\beta = l/2$. The same logic applies to the other bead, so that $\mathbf{L}$ for the beads does not change once they leave the rod. This in turn means that the remaining angular momentum, the angular momentum possessed by the rod, does not change either, and since the moment of inertia of the rod stays put its angular velocity ω must remain constant, too.



(iii) While the beads are on the rod, the force of friction $\mathbf{F}$ acts in the same direction as the position $\mathbf{r}$, so the torques are $\mathbf{r} \times \mathbf{F} = 0$. But it is torques, and not forces, that affect rotational motion, so $\mathbf{F}$ can have no effect. This applies to when the beads are on the rod; once they leave the rod, there is of course no friction.

8. A sphere of radius R is supported by a rope attached to a wall as shown in Fig. 4. The rope makes an angle θ with respect to the wall. The point where the rope is attached to the ball is such that if the line of the rope is extended it crosses the horizontal line through the center of the ball at a distance $3R/2$ from the wall. Show that the minimum μ_s between wall and ball for this to be

possible is $\mu_s = \frac{1}{2}\cot\theta$. Evaluate this for $\theta = 30°$. Hint: Find the right place to take torques. The usual suspects will not do.

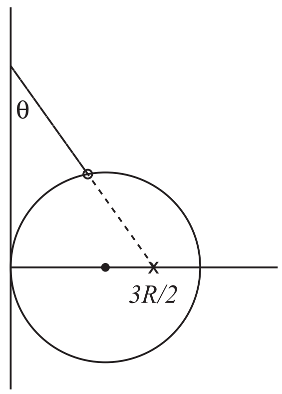
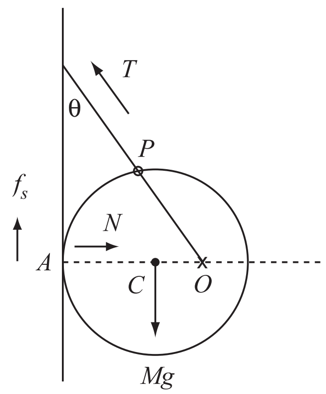


Figure 4: The rope is attached to the wall at one end and the ball at the other. If extrapolated, the line of the rope crosses the horizontal line going through the center at a distance $3R/2$ from the wall.

Answer: We need to balance both the forces and the torques, and it is the latter that is tricky here. So consider the figure below:



In order to make calculating the torque due to the rope simple, we should choose point O as the point around which we take torques. Then the tension in the rope has no component perpendicular to OP and hence applies no torque in our system. The forces that do apply a torque are the force of gravity acting downward at C, so that $\tau_g = MgR/2$, and the force of friction acting upward at A, so that $\tau_f = 3Rf_s/2$, where f_s is the force of static friction. These torques act in opposite directions, so $\tau_g = \tau_f$ implies

$$\frac{MgR}{2} = \frac{3Rf_s}{2} \implies f_s = \frac{Mg}{3}.$$

We can now use our old tools to find f_s. In the horizontal direction, the normal force N balances the horizontal component of the tension $T\sin\theta$, so $N = T\sin\theta$. In the vertical direction, the vertical component of the tension $T\cos\theta$ and force of friction $f_s = \mu_s N$ balance the force of gravity Mg:

$$T\cos\theta + f_s = Mg.$$

Substituting $T = N/\sin\theta = f_s/\mu_s \sin\theta$, we get

$$\left(1 + \frac{\cot\theta}{\mu_s}\right) f_s = \left(1 + \frac{\cot\theta}{\mu_s}\right) \cdot \frac{Mg}{3} = Mg,$$

or

$$1 + \frac{\cot\theta}{\mu_s} = 3 \implies \mu_s = \frac{\cot\theta}{2} = \frac{\sqrt{3}}{2} = 0.87$$

if $\theta = 30°$.

9. A horizontal rod of mass 8 kg and length 2.4 m is hinged to a wall and supported by a cable that makes an angle of 25° as shown in Fig. 5. What is the tension T on the cable and what is the force exerted by the pivot? Repeat if in addition a 25 kg weight is suspended at the end of the rod.

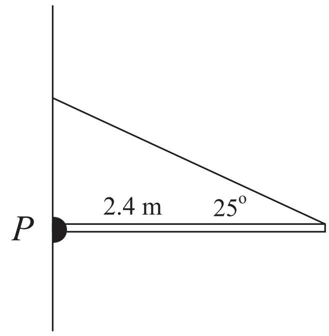
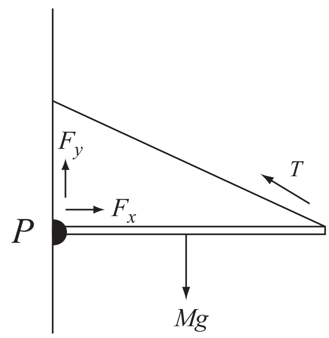


Figure 5: The rope is attached to the wall at one end and the rod at the other. P is the pivot.

Answer: If we choose the pivot as the center of torque, then the nonzero torques acting on the rod are due to gravity, $\tau_g = Mgl/2$, and the vertical component of the tension, $\tau_r = Tl\sin\theta$. These act in opposite directions and balance each other, so that

$$\frac{Mgl}{2} = Tl\sin\theta \implies T = \frac{Mg}{2\sin\theta} = \frac{(8\text{ kg})(9.81\text{ m/s}^2)}{2\sin 25°} = 93\text{ N}.$$



Since the tension has a leftward component and gravity acts entirely downward, the pivot must exert a force to the right, $F_x = T\cos\theta = \frac{Mg}{2}\cot\theta$, to balance the tension. In the vertical direction, we have

$$F_y + T\sin\theta = F_y + \frac{Mg}{2} = Mg \implies F_y = \frac{Mg}{2}.$$

Therefore the force exerted by the pivot is

$$\mathbf{F} = \frac{Mg}{2}(\cot\theta\hat{\mathbf{x}} + \hat{\mathbf{y}}) = \frac{(8\text{ kg})(9.81\text{ m/s}^2)}{2}(\cot 25^\circ\hat{\mathbf{x}} + \hat{\mathbf{y}}) = (39\text{ N})(2.1\hat{\mathbf{x}} + \hat{\mathbf{y}}) = (84\text{ N})\hat{\mathbf{x}} + (39\text{ N})\hat{\mathbf{y}}.$$

If, in addition, an object of mass m is attached to the end of the rod, there is an additional torque mgl that counters the torque due to tension:

$$T'l\sin\theta = \frac{Mgl}{2} + mgl \implies T' = \frac{g(M/2+m)}{\sin\theta} = \frac{(9.81\text{ m/s}^2)[(8\text{ kg})/2 + 25\text{ kg}]}{\sin 25^\circ} = 670\text{ N}.$$

The horizontal force is again given by

$$F'_x = T'\cos\theta = g\left(\frac{M}{2} + m\right)\cot\theta,$$

while the vertical force is given by

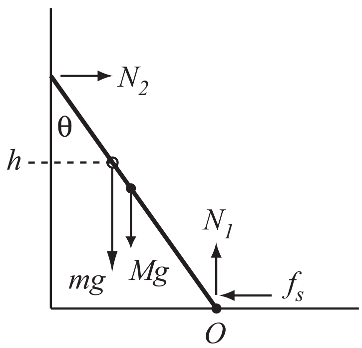
$$F'_y + T'\sin\theta = F'_y + g\left(\frac{M}{2} + m\right) = (M+m)g \implies F'_y = \frac{Mg}{2}.$$

Therefore

$$\begin{aligned}\mathbf{F}' &= \left(\frac{M}{2} + m\right)g\cot\theta\hat{\mathbf{x}} + \frac{Mg}{2}\hat{\mathbf{y}} = \left(\frac{8\text{ kg}}{2} + 25\text{ kg}\right)(9.81\text{ m/s}^2)\cot 25^\circ\hat{\mathbf{x}} + \frac{(8\text{ kg})(9.81\text{ m/s}^2)}{2}\hat{\mathbf{y}}\\ &= (610\text{ N})\hat{\mathbf{x}} + (39\text{ N})\hat{\mathbf{y}}.\end{aligned}$$

10. A ladder of length 6 m mass 15 kg leans against a wall at angle 30° *with respect to the wall.* With respect to the ground it has $\mu_s = 0.4$. How high can a 70 kg man climb before the ladder slips?

Answer: The forces in the problem are shown below, where we call the mass of the ladder M and the mass of the man m. Suppose the man is standing at height h. Since there are two forces acting at the point where the ladder meets the floor, we can get rid of some torques by choosing this point as the origin.



The nonzero torques due to N_2, Mg, and mg balance:

$$N_2 l\cos\theta = \frac{1}{2}Mgl\sin\theta + \frac{mgh}{\cos\theta}\sin\theta \implies h = \frac{N_2 l\cos^2\theta}{mg\sin\theta} - \frac{Ml\cos\theta}{2m}.$$

From the balance of forces in the vertical direction we get $N_1 = (M+m)g$, and from the horizontal direction we get (actually, what we want is $N_2 \leq f_s$ so the ladder doesn't slip, which is why h is the maximum value for which the ladder will not slide)

$$N_2 = f_s = \mu_s N_1 = \mu_s g(M+m).$$

Plugging into the above expression for h gives

$$\begin{aligned} h &= l\cos\theta\left[\mu_s\left(1+\frac{M}{m}\right)\cot\theta - \frac{M}{2m}\right] \\ &= (6\text{ m})\cos 30^\circ\left[0.4\left(1+\frac{15\text{ kg}}{70\text{ kg}}\right)\cot 30^\circ - \frac{15\text{ kg}}{2(70\text{ kg})}\right] = 3.8\text{ m}. \end{aligned}$$