Midterm Solutions

(I) *A bullet of mass m moving at horizontal velocity v strikes and sticks to the rim of a wheel (a solid disc) of mass M, radius R, anchored at its center but free to rotate. (i) Which of energy, momentum and angular momentum is conserved for the bullet+wheel system? Give a few words of explanation. (ii) Find ω_f the final angular velocity of the wheel.* **10**

(i) Only angular momentum is conserved. The collision is an inelastic collision, so energy cannot be conserved. Momentum is not conserved because there is an external force acting at the center of the wheel that keeps the wheel-bullet system from moving forward after the collision. Angular momentum is conserved because there are no external torques acting on the system. (The force acting at the center of the wheel does not provide a torque because it is acting at the pivot point.)

(ii) To find ω_f, we equate the angular momentum of the system just before the bullet hits the wheel with the final angular momentum of the rotating bullet-wheel system.

$$\begin{aligned} mvR &= \omega_f (I_{wheel} + I_{bullet}) \\ &= \omega_f \left(\frac{1}{2}MR^2 + mR^2\right) \\ \omega_f &= \frac{v}{R}\left(\frac{1}{1+\frac{M}{2m}}\right). \end{aligned}$$

Please note that linear momentum p and angular momentum L are *not* equivalent. They have separate conservations laws, different units, and measure fundamentally different quantities. You cannot equate p to L.

(II) *A block of mass M sits on frictionless table L meters from the edge. At $t = 0$ bullet of mass m and velocity v_1 penetrates it from left and exits to the right with a speed v_2. (i) When will the block fly off the table? (ii) If the table has a height h how far from the edge of the table will it land?* **10** *(Neglect loss of wood in block due to bullet penetrating it and the time it takes bullet to traverse block.)*

(i) To find when the block will fly off the table, we need to calculate the final velocity of the block by conserving momentum before and after the collision.

$$mv_1 = mv_2 + Mv \implies v = \frac{m}{M}(v_1 - v_2)$$

There are no external forces on the block, so the block will fly off the table at

$$t = \frac{L}{v} = \frac{M}{m}\left(\frac{L}{v_1 - v_2}\right).$$

(ii) To find the distance the block travels in the x direction after leaving the table, we first need to find the total time it spends in the air. We know that the block falls a distance h in the y direction and has no initial velocity in y. This tells us that

$$h = \frac{1}{2}gt^2 \implies t = \sqrt{\frac{2h}{g}}$$

(clearly, the negative solution for t is not physical for this problem). There are no forces in the x direction, so the total distance the block travels once it leaves the table is

$$x = vt = (v_1 - v_2)\frac{m}{M}\sqrt{\frac{2h}{g}}.$$

(III) *Consider the force $\mathbf{F} = \mathbf{i}2xy^3 + \mathbf{j}3x^2y^2$. (i) Show that it is conservative. (ii) What is the potential energy $U(x, y)$ associated with it ? (iii) What is the work done by the force along a path $y = x^{123456789}$ joining $(0, 0)$ to $(1, 1)$?* **10**

(i) A force given by $\mathbf{F} = \mathbf{i}F_x + \mathbf{j}F_y$ is conservative if $\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$. This is because in order for a force to be conservative, it must be described by a potential that satisfies $\frac{\partial U}{\partial x \partial y} = \frac{\partial U}{\partial y \partial x}$ where $U = -\int \mathbf{F} \cdot d\mathbf{x}$. In this case, the condition for $\mathbf{F}$ to be conservative gives

$$\frac{\partial F_x}{\partial y} = 6xy^2 \quad \text{and} \quad \frac{\partial F_y}{\partial x} = 6xy^2,$$

so $\mathbf{F}$ is conservative.

(ii) We know that

$$\frac{\partial U}{\partial x} = -F_x \quad \text{and} \quad \frac{\partial U}{\partial y} = -F_y,$$

The only function that satisfies these conditions (up to a constant) is $U(x, y) = -x^2y^3$.

(iii) The work done between points 1 and 2 for a conservative force is

$$\begin{aligned}\int_1^2 \mathbf{F} \cdot d\mathbf{x} &= -U(2) - (-U(1)) \\ &= U(1) - U(2) \\ &= U(0,0) - U(1,1) \\ &= -(-1) \\ &= 1\end{aligned}$$

You could have also integrated along the path given in the problem or any other path you wanted to get the correct answer. Since the force is conservative, the answer is path independent.

(IV) *(i) Why can a body with total energy $E < 0$ not escape to infinity?* **3**

For a body bound in a gravitational potential,

$$E = \frac{1}{2}mv^2 - \frac{GMm}{R}.$$

If $R = \infty$ then the potential energy term is zero and E will necessarily be greater than or equal to zero. Hence, for a mass with $E < 0$, R must always have a finite value and therefore the mass can never escape to infinity. (Just saying that a body must have $E > 0$ to escape from a gravitational potential is not sufficient because you just restated the question. You must explain why that is the case.)

(ii) I give you a spring of unknown force constant k, a meter stick , a clock, a 1kg mass and a block of wood at rest on a table. How will you find μ_s, the coefficient of static friction between the block and the table. (Help yourself to my tool box with massless hooks, nails etc.) **7**

The coefficient of static friction measures the maximum frictional force available while the block is *not* moving. (If I measured the force due to friction of a moving block, this would give the coefficient of *kinetic* friction.) The experiment that I want to do to find μ_s is to place the block of wood on the table and attach the spring between the block and a wall (see Figure 1a). I will then measure the maximum distance I can pull the block away from the wall without it being pulled back by the spring. This maximum distance occurs when the force exerted by the spring just balances the maximum frictional force. Take the mass of the block to be M, so $N = Mg$ on a flat surface. Balancing the forces we have,

$$Ma = 0 = -kA + \mu_s N \implies \mu_s = \frac{kA}{Mg} \tag{1}$$

where A is how far I can pull the mass before it just starts to move.

To determine μ_s from this equation, I need to measure M, and k. I can find k by hanging the $1\,kg$ mass (m) from the spring and measuring the spring's displacement x_m (see Figure 1b). Balancing the force due to gravity with the force from the spring

$$mg = kx_m \implies k = \frac{mg}{x_m}.$$

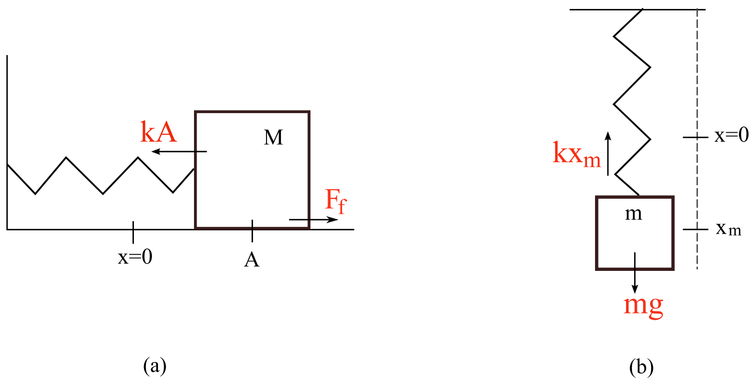

FIG. 1: Experiments for problem 4 for part (ii).

Now, I can repeat this experiment, but using the block M. Plugging in for k this gives,

$$Mg = kx_b = \frac{mg}{x_m}x_b \implies M = m\frac{x_b}{x_m}.$$

Now we can plug back into equation 1 for μ_s

$$\begin{aligned}\mu_s &= \frac{kA}{Mg} \\ &= \frac{A}{g}\frac{mg}{x_m}\frac{x_m}{mx_b} \\ \mu_s &= \frac{A}{x_b}.\end{aligned}$$

So, we really only need to do two experiments to determine μ_s. We need to hang the block from the spring under the influence of gravity and measure how far the spring stretches. This is x_b. We then need to attach the spring to a wall such that the block can sit on the table and measure how far you can pull the block away from the wall before the spring can begin to pull it back. This gives us A. The coefficient of static friction is then just A/x_b.

(iii) I give you two spheres of same mass M and radius R, one solid and one hollow, and an incline on which they can roll without slipping. Explain how you will say which is which, giving me some idea you understand why your suggestion will work. **5**

By conservation of energy,

$$\begin{aligned}Mgh &= \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \\ &= \frac{1}{2}M(R\omega)^2 + \frac{1}{2}I\omega^2\end{aligned}$$

If you start both spheres rolling at the same heigh h, they will have the same energy at the bottom of the inclined plane. Since the energy must be the same, it is clear that which ever sphere has a larger moment of inertia, I, will have a smaller value of ω. The hollow sphere will have a larger moment of inertia because all of its mass is located a distance R from the center. The solid sphere has its mass distributed between $r = 0$ and $r = R$. So, if you roll both of the spheres down the inclined plane starting from the same heigh, the hollow sphere will be the one moving more slowly at the bottom.

(V) *A mass m tethered to a massless string is spinning in a* vertical circle, *keeping its total energy constant. Find the difference in the (magnitude of) the tension between the top most and bottom most points.* **10**

When the mass is at the top of the circle, both the tension from the string and gravity point down, as shown in Figure 2. Equating these forces with the force required to keep the mass moving in a circle we find

$$m\frac{v_t^2}{R} = mg + T_t \implies T_t = m(\frac{v_t^2}{R} - g)$$

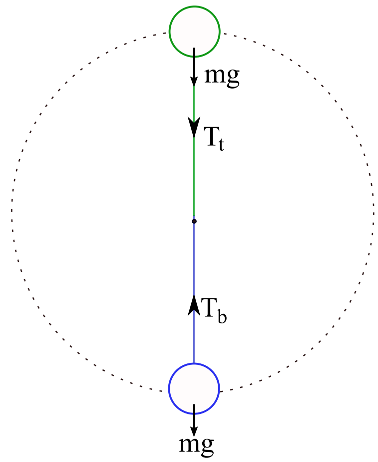

FIG. 2: Figure for problem 5 that shows the forces on the ball ball both at the top and bottom of the circle. The angular acceleration, v^2/R, always points towards the center.

where T_t is the tension at the top and v_t is the velocity at the top. When the mass is at the bottom of the circle, the tension points up towards the center of the circle, while gravity acts down. Again, balancing the forces,

$$m\frac{v_b^2}{R} = -mg + T_b \implies T_b = m(\frac{v_b^2}{R} + g).$$

The difference between T_b and T_t is

$$|T_b| - |T_t| = m\left(\frac{v_b^2 - v_t^2}{R} + 2g\right). \tag{2}$$

Now we need to find an equation for $v_b^2 - v_t^2$. The other piece of information that we are given is that the energy is constant. Equating the energy at the top and bottom of the circle we find

$$\frac{1}{2}mv_b^2 = \frac{1}{2}mv_t^2 + 2mgR$$
$$4gR = v_b^2 - v_t^2$$

Plugging this result into equation 2 we find that the difference in the tension is

$$|T_b| - |T_t| = m(4g + 2g) = 6mg.$$

(VI) *A horizontal rod of length L and mass M has a mass m at one end. It is supported by pivot P on the wall at the left end and a cable at angle of θ at the other end as shown in Figure 3.*
(i) Find T, the tension on the cable.
(ii) If the cable snaps, with what angular velocity ω will the rod swing down and slam into the wall? **15**

(i) In this part of the problem everything is stationary, so the sum of the torques must equal zero. As shown in Figure 3, there are three forces in this problem that are not acting at the pivot P, the tension in the cable, the force of gravity from mass m, and the force of gravity from the rod M. Calculating the torques about the pivot P,

$$\sum \tau = 0 = Mg(L/2) + mgL - TL\sin\theta$$
$$T = \frac{g}{\sin\theta}\left(\frac{M}{2} + m\right).$$

(ii) If the cable snaps, we can conserve energy to find the angular velocity when the rod hits the wall. Just when the cable snaps, the rod is at rest, so the only energy is potential energy. If we take the distance L below the pivot to have zero potential energy, then initially, the energy of the system is

$$E_i = (M + m)gL.$$

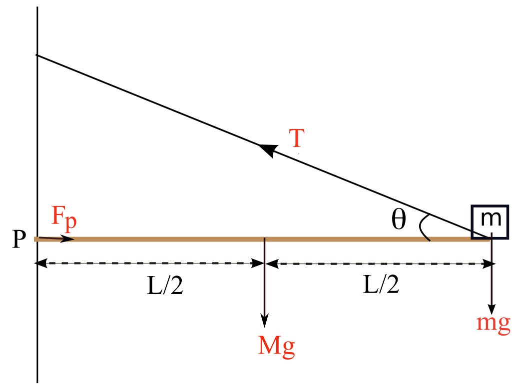

FIG. 3: The forces on the rod in problem 6.

When the rod reaches the wall, the rod-mass system will have some angular velocity ω and the center of mass of the rod will be at a height $L/2$, so the final energy is

$$E_f = Mg\frac{L}{2} + \frac{1}{2}I_m\omega^2 + \frac{1}{2}I_{rod}\omega^2.$$

The mass m is a distance L from the pivot, so $I_m = mL^2$. A rod has a moment of inertia $I = ML^2/12$ about its center of mass. Using the parallel axis theorem, the moment of inertia of the rod about the pivot is $I_{rod} = ML^2/12 + M(L/2)^2 = ML^2/3$. Plugging in the moments of inertia and setting the initial and final energies equal,

$$\begin{aligned}
gL(M+m) &= Mg\frac{L}{2} + \frac{1}{2}\left(mL^2\right)\omega^2 + \frac{1}{2}\left(\frac{1}{3}ML^2\right)\omega^2 \\
g\left(m + \frac{M}{2}\right) &= \omega^2\left(\frac{mL}{2} + \frac{ML}{6}\right) \\
\omega^2 &= \frac{g}{L}\left(\frac{2m+M}{m+M/3}\right) \\
\omega &= \sqrt{\frac{g}{L}\left(\frac{2m+M}{m+M/3}\right)}.
\end{aligned}$$

Alternatively, you can do this problem using torques and acceleration. This is significantly more complicated, and you don't need to know how to do this; however, many of you tried, so this is the solution. The torque on the beam is $\tau = I\alpha = \alpha(mL^2 + \frac{1}{3}ML^2)$. You can also calculate the torque using forces, but note that as the beam falls, the component of the force acting perpendicular to the beam changes.

$$\tau = mgL\sin\phi + Mg\frac{L}{2}sin\phi$$

where ϕ is the angle the beam makes with the wall. Now, we can equate these two equations for torque and recognize that α is the second derivative of ϕ with respect to time or $\alpha = \ddot{\phi}$. We find that,

$$g\sin\phi\left(m + \frac{M}{2}\right) = L\left(m + \frac{1}{3}M\right)\ddot{\phi}. \tag{3}$$

Now, I will integrate this equation with respect to ϕ from $\phi = \pi/2$ to $\phi = \pi$. This is easy for the left hand side of Equation 3, but a little more complicated for the right hand side. On the right hand side, I want to integrate with respect to time because that will give me $\dot{\phi} = \omega$. Multiplying $d\phi$ by dt/dt, $d\phi = \frac{d\phi}{dt}dt = \dot{\phi}dt$. Carrying out the integration,

$$\int_{\pi/2}^{\pi} g \sin\phi \left(m + \frac{M}{2}\right) d\phi = \int_0^{t_f} L \left(m + \frac{1}{3}M\right) \ddot{\phi}\dot{\phi} dt$$

$$-\frac{g}{L}\left(m + \frac{M}{2}\right)(\cos(\pi) - \cos(\pi/2)) = \left(m + \frac{1}{3}M\right)\frac{\dot{\phi}^2}{2}\Big|_{t=0}^{t=t_f}$$

$$\frac{g}{L}\left(m + \frac{M}{2}\right) = \left(m + \frac{1}{3}M\right)\frac{\omega_f^2}{2}$$

$$\omega_f^2 = \frac{g}{L}\left(\frac{2m + M}{m + M/3}\right)$$

which is the same answer we found above using energy conservation.